Stable commutator length in graphs of groups

Lvzhou Chen

NCNGT
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Stable commutator length (scl)

Definition

Space X, null-homologous loop γ. Then γ bounds a surface.

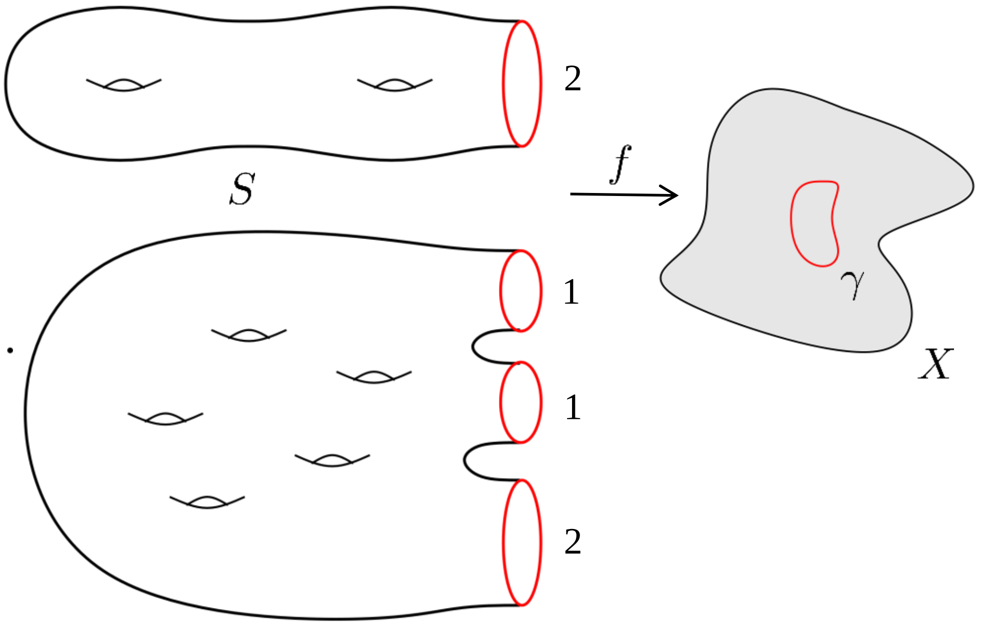
Admissible surfaces (S, f)

$$\mathrm{scl}_X(\gamma) := \inf_{(S,f)} \frac{-\chi^-(S)}{2\cdot n}$$

Multiplicative!

$$\chi^-(S) = \chi(S - D^2\text{'s } S^2\text{'s}).$$

$$\mathrm{scl}_G(g) = \mathrm{scl}_X(\gamma)$$



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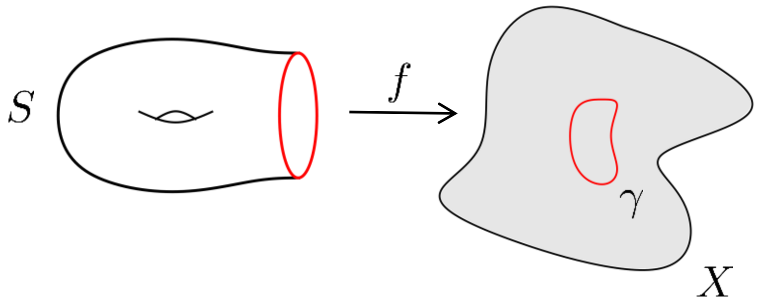

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Example

$\mathrm{scl}_G([x, y]) \leq 1/2$.

Space of surfaces has little structure.

Challenge: prove lower bounds of scl or compute it.

Properties

- algebraic: only depends on $G = \pi_1(X)$,
and has a group-theoretic definition using commutators.
- non-increasing: $\forall \varphi : G \to H$ homomorphism, $\mathrm{scl}_G(g) \geq \mathrm{scl}_H(\varphi(g))$.
- characteristic: $\forall \varphi \in \mathrm{Aut}(G)$, $\mathrm{scl}_G(\varphi(g)) = \mathrm{scl}_G(g)$.
- dual to quasimorphisms, related to $H_b^2(G)$.
- $\mathrm{scl}_G \equiv 0$ if G is amenable or a higher rank irreducible lattice.
- non-trivial on word-hyperbolic groups, mapping class groups, etc.

Gromov–Thurston norm

Gromov Norm

$A \subset X$ subspace, $\alpha \in H_2(X, A; \mathbb{Z})$. $(S, \partial S) \to (X, A)$ represents α, the *Gromov norm*

$$\|\alpha\| := \inf_{[S]=n\alpha} \frac{-2\chi^-(S)}{n}.$$

When $(X, A) = (M^3, \partial M)$,

$$\|\alpha\|_{Th} := \inf_{\substack{[S]=\alpha \\ \textit{embedded}}} -\chi^-(S) = \frac{1}{2}\|\alpha\|.$$

Remove "embedded" by Gabai–Thurston

Example

$K \subset S^3$ knot, $X = S^3 \setminus N(K)$, $A = \partial X$, $\gamma = K = \partial\Sigma$, then there is a unique $\alpha \in H_2(X, A)$ with $\partial\alpha = [\gamma]$. We have

$$\mathrm{scl}_G(\gamma) = \tfrac{1}{4}\|\alpha\| = \tfrac{1}{2}\|\alpha\|_{Th} = g(K) - \tfrac{1}{2} \in \mathbb{Q}$$

extremal

$$0 = H_2(X) \longrightarrow H_2(X, A) \longrightarrow H_1(A) \longrightarrow H_1(X)$$

Extremal surfaces

Definition

An admissible surface (S, f) is *extremal* for g if

$$\mathrm{scl}_G(g) = \inf_{(S,f)} \frac{-\chi^-(S)}{2 \cdot n}$$

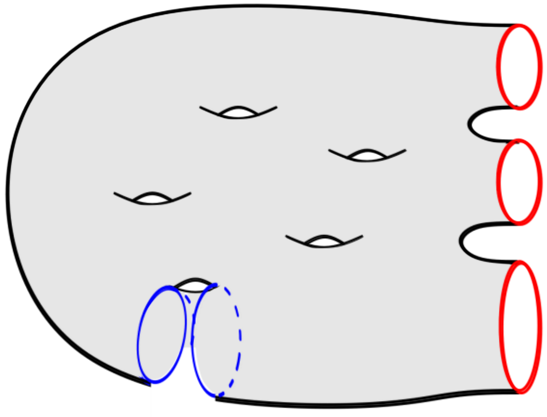
Note: extremal $\xrightarrow{\text{finite cover}}$ extremal

Existence $\implies \mathrm{scl}_G(g) \in \mathbb{Q}$

Proposition

If (S, f) is extremal then $f_* : \pi_1(S) \to G$ is injective.

Proof: Compression+LERF



Computations and rationality

Theorem (Calegari'08)

$G = F_n$, there is a linear programming algorithm to compute $\mathrm{scl}_G(g)$ and produces *extremal surfaces* (in particular $\mathrm{scl}_G(g) \in \mathbb{Q}$).

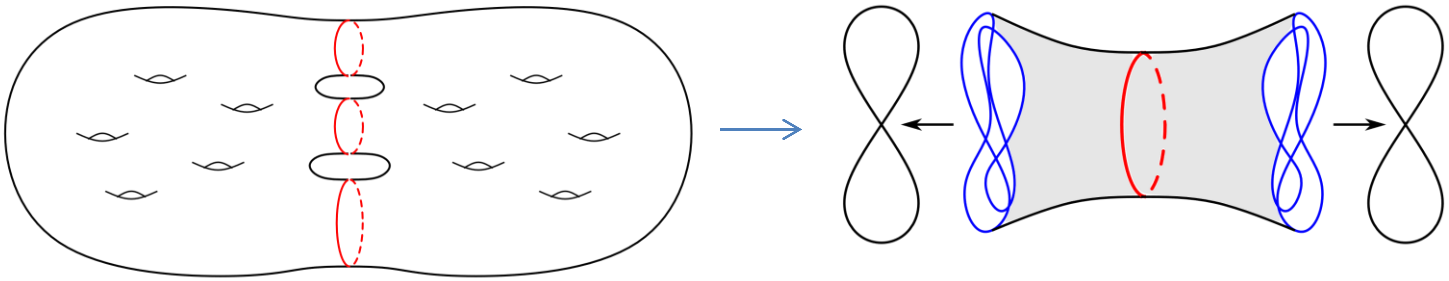
Gromov's Question

Does every one-ended word-hyperbolic group G contain a $\pi_1(S)$?

- Kahn–Marković'09: yes if $G = \pi_1(M)$, M^3 closed hyperbolic.

Application (surface subgroups)

Calegari'08: $g \in [F_n, F_n]$, $G = DF_n(g)$ has a surface subgroup.



Main theorem

Theorem (C.'19)

G a graph of groups, such that for each vertex group G_v f.i

(1) $\mathrm{scl}_{G_v} \equiv 0$; and (e.g. G_v amenable, $\mathrm{SL}_3\mathbb{Z}$) $H, K \le G$, $H \cap K \le H, K$

(2) $\mathrm{Im}(G_e \to G_v)$ are central and mutually commensurable;

Then there is a linear programming algorithm to compute $\mathrm{scl}_G(g)$, which is rational.

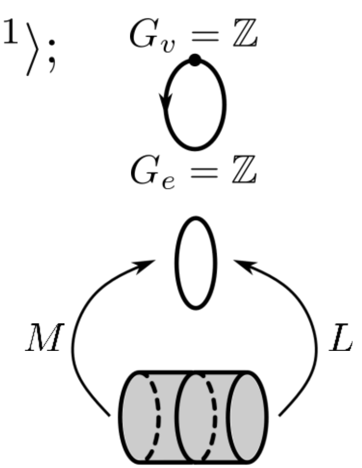
Examples

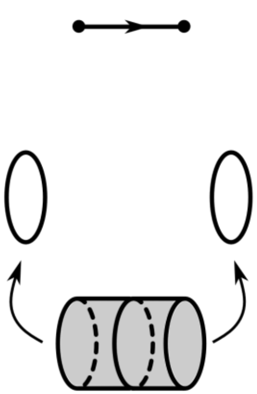
$G = \mathrm{BS}(M, L) := \langle a, t \mid a^M = t a^L t^{-1} \rangle$;

(Clay–Forester–Louwsma'13): alternating words in $\mathrm{BS}(M, L)$;

(C.'16) $G = *_\lambda G_\lambda$ with $\mathrm{scl}_{G_\lambda} \equiv 0$;

(Susse'13) $G = \mathbb{Z}^m *_{\mathbb{Z}^k} \mathbb{Z}^n$.





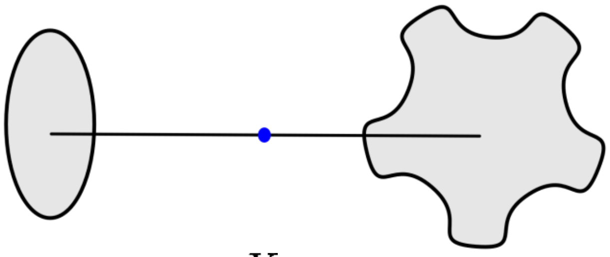
Proof ideas

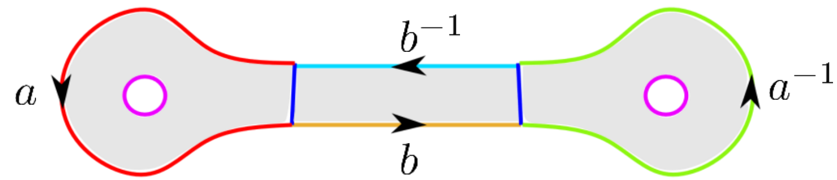
- **Step 1**: Obtain simple normal form of (relative) admissible surfaces.
  Each is made of simple "Lego" pieces.

- **Step 2**: Use linear programming to find the best combination of
  "Lego" pieces.

Relative admissible surfaces

- relative admissible surface: allow extra boundary curves in vertex spaces.



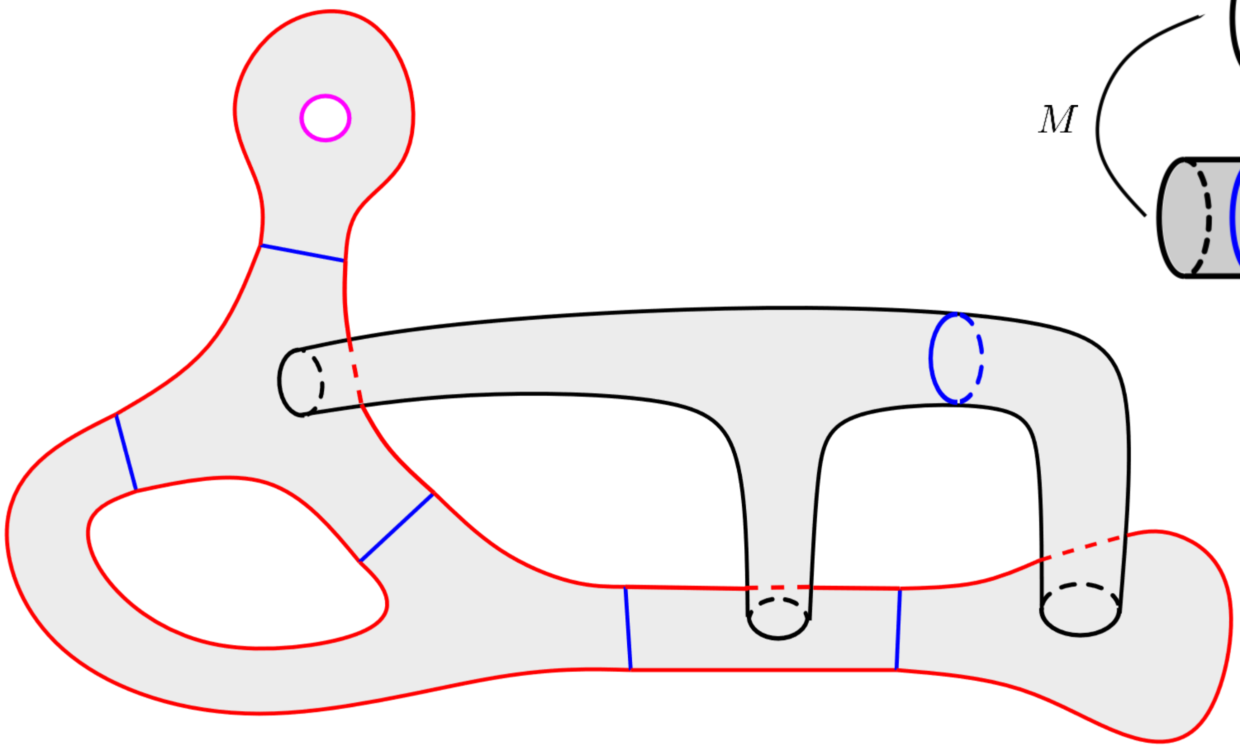


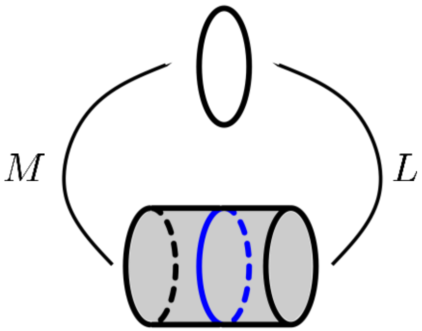
- **Lemma**: $\mathrm{scl}_G(g) = \inf\limits_{\substack{(S,f)\\ \text{adm}}} \frac{-\chi^-(S)}{2\cdot n} \geq \inf\limits_{\substack{(S,f)\\ \text{rel adm}}} \frac{-\chi^-(S)}{2\cdot n}$.

  "$=$" if $\mathrm{scl}_{G_v} \equiv 0$

Simple normal form

- Edge spaces cut S into subsurfaces supported in vertex spaces.
- Simplify each component into a disk or annuli.





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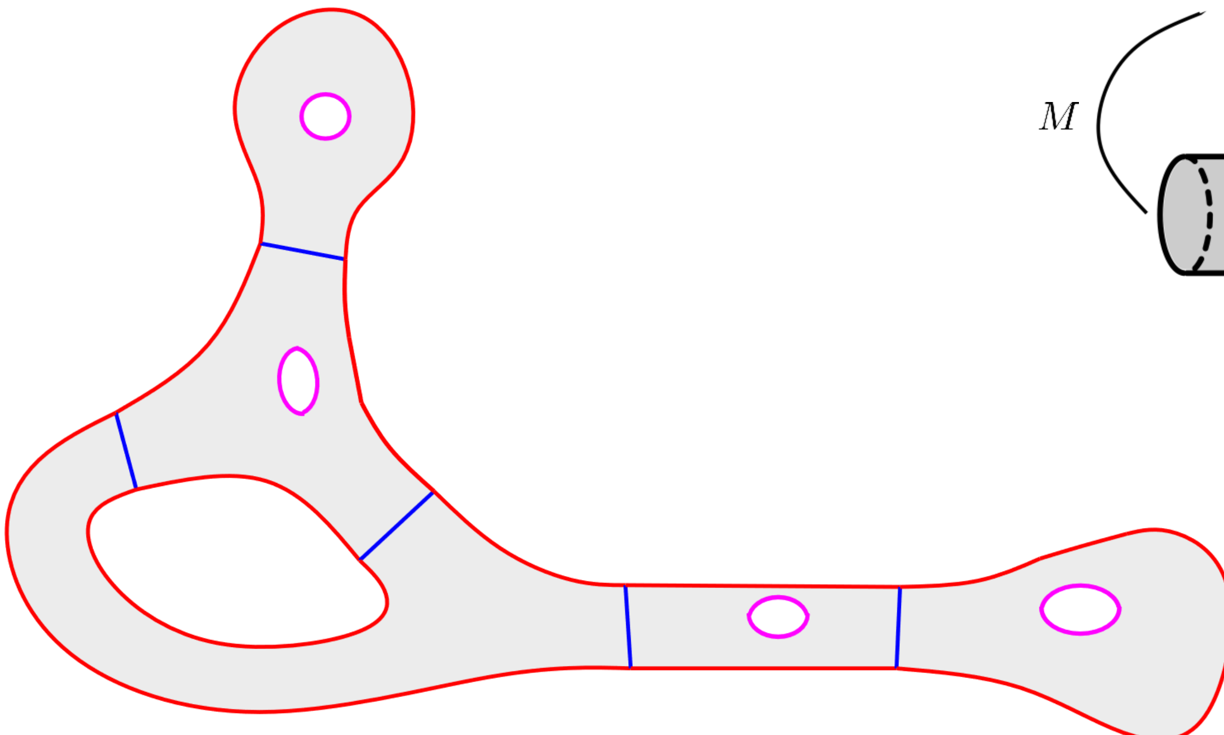
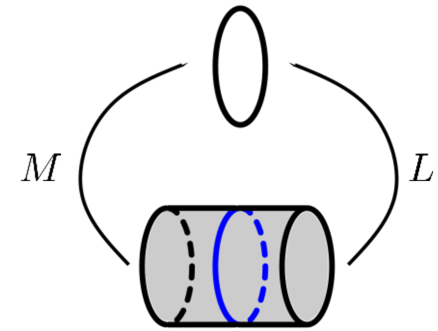


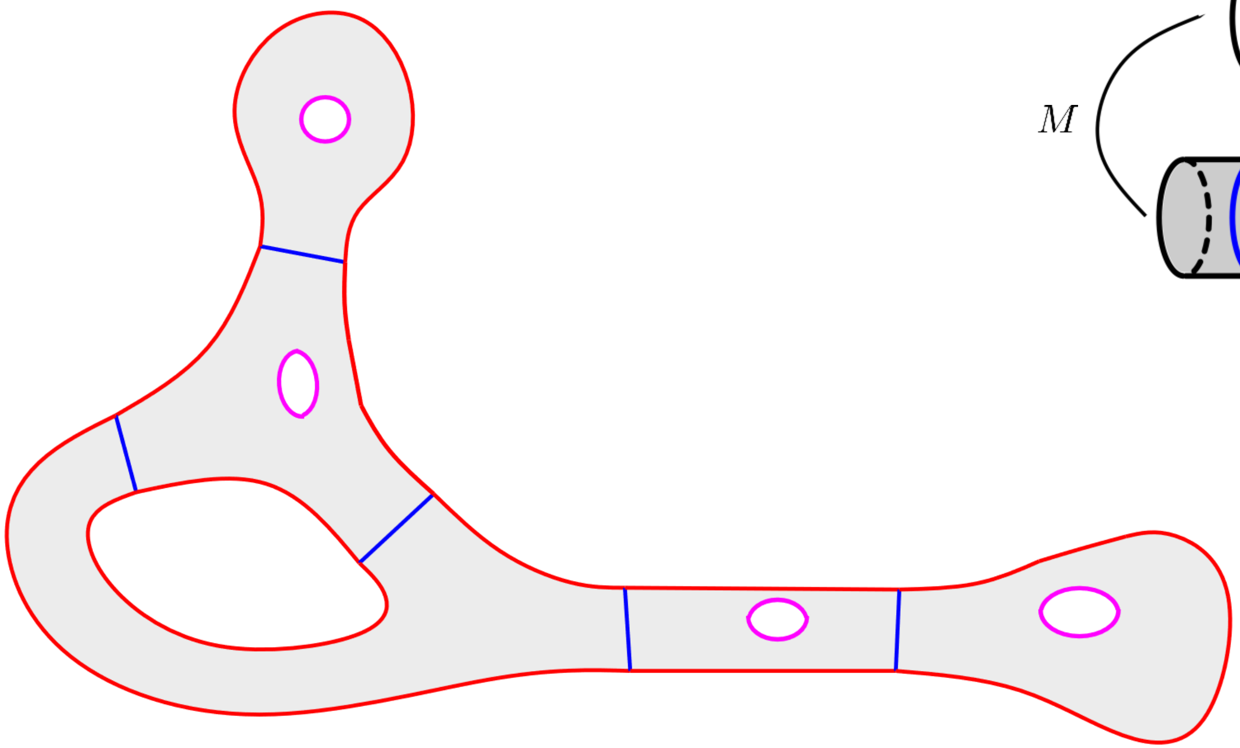


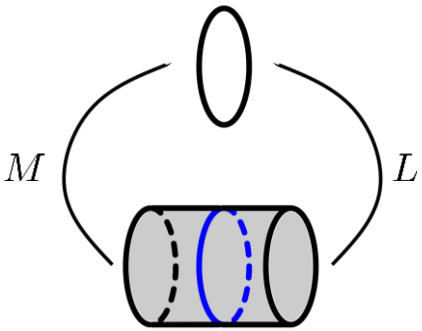
- Disk iff winding number $= 0$.

Linear programming

minimize $\frac{-\chi(S)}{2n}$ $\iff$ maximize #disk pieces$/n$

max: $\sum x_D$
subj: gluing & normalizing conditions





Dimension reduction (BS case)

Fix a suitable $D \in \mathbb{Z}_+$.

- A piece is disk-like if its winding number is divisible by D.
- $\widehat{\chi}(S) = \chi(S) + \#\{\text{disk-like but not disk}\}$.

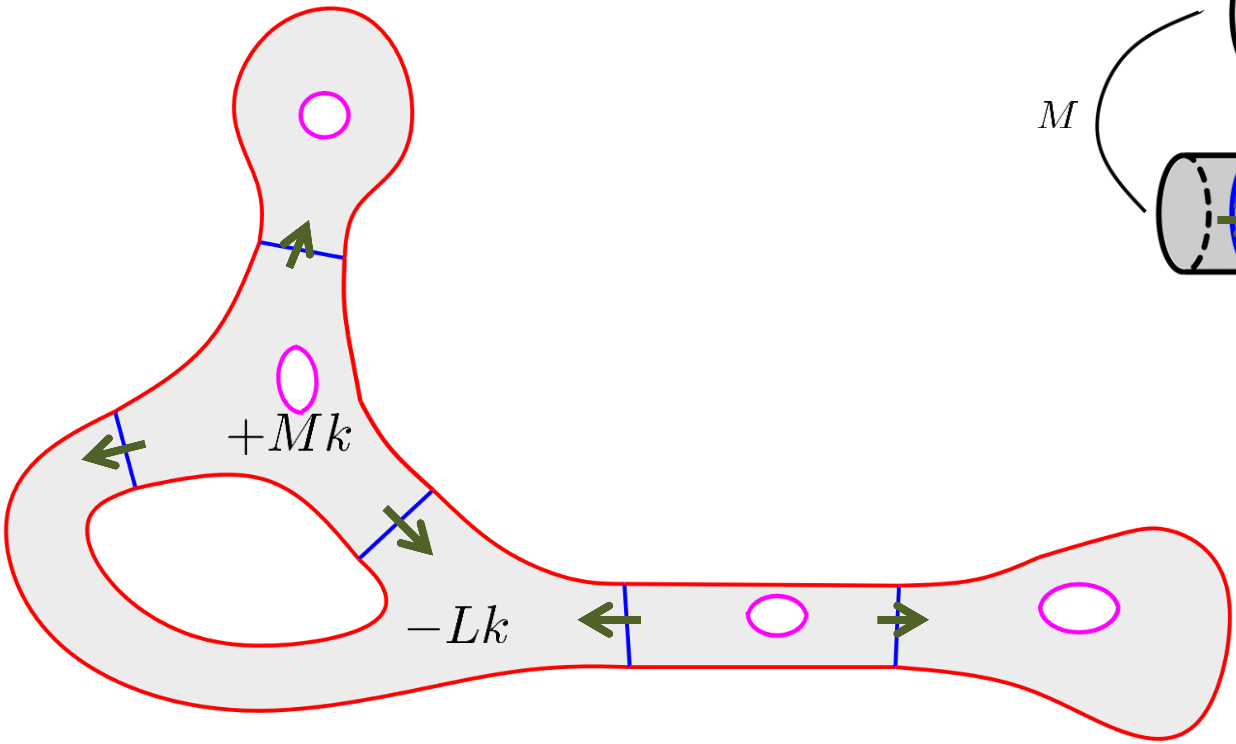
$$\Longrightarrow \inf_{\substack{S \\ \text{rel adm}}} \frac{-\widehat{\chi}(S)}{2n(S)} \leq \inf_{\substack{S \\ \text{rel adm}}} \frac{-\chi(S)}{2n(S)} = \mathrm{scl}(\gamma).$$

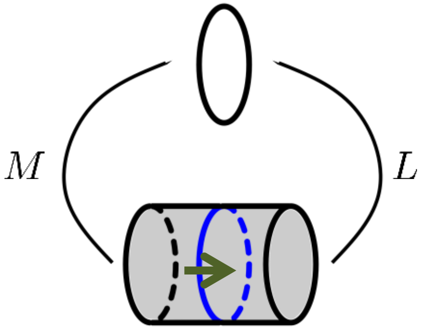
Key Lemma: There is a suitable $D = D(\gamma)$ such that "$=$" holds: for any relative admissible S and any $\epsilon > 0$, there is another $\widehat{S}$ satisfying

$$\frac{-\chi(\widehat{S})}{2n(\widehat{S})} < \frac{-\widehat{\chi}(S)}{2n(S)} + \epsilon.$$

Asymptotic promotion

- there is $N = N(\gamma)$ such that if $dm^N \ell^N \mid w(C)$, $(M = dm,\ L = d\ell)$ then C can be promoted to a disk asymptotically.





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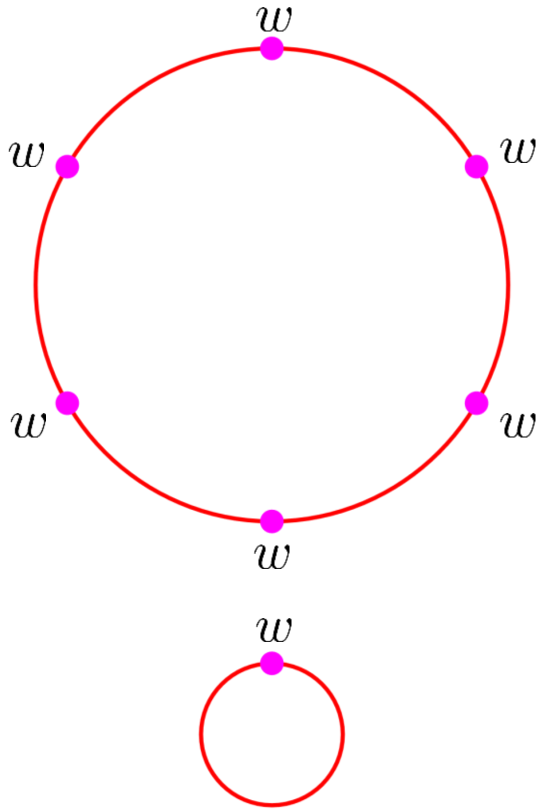

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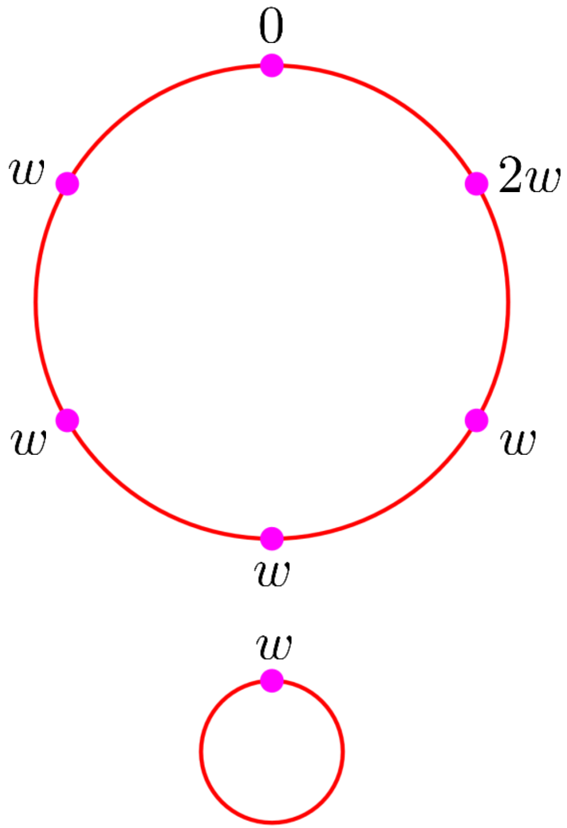

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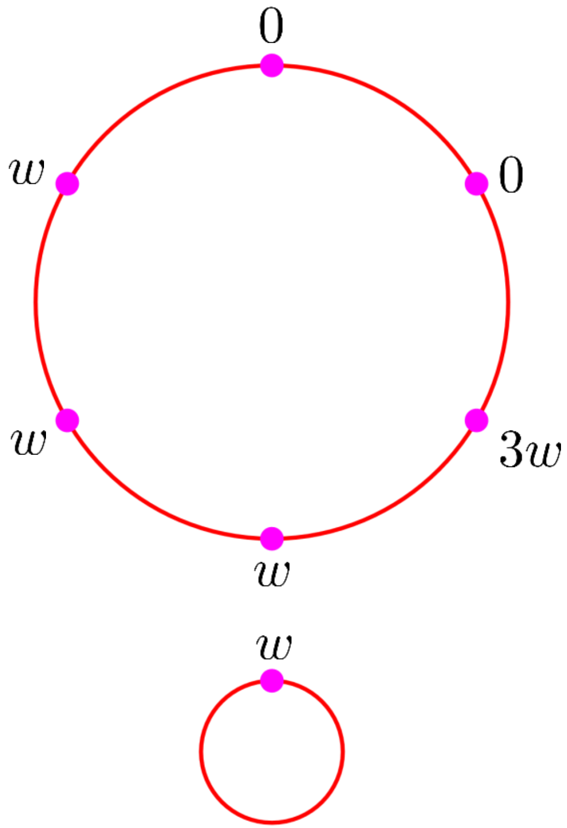

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