THE WIEGOLD PROBLEM AND FREE PRODUCTS OF LEFT-ORDERABLE GROUPS

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Abstract. A group has normal rank (or weight) greater than one if no single element normally generates the group. The Wiegold problem from 1976 asks about the existence of a finitely generated perfect group of normal rank greater than one. We show that any free product of nontrivial left-orderable groups has normal rank greater than one. This solves the Wiegold problem by taking free products of finitely generated perfect left-orderable groups, a plethora of which are known to exist. We obtain our estimate of normal rank by a topological argument, proving a type of spectral gap property for an unsigned version of stable commutator length. A key ingredient in the proof is an intricate new construction of a family of left-orders on free products of two left-orderable groups.

1. Introduction

A fundamental concept in group theory is the *normal rank* (or *weight*) of a group, which is the smallest cardinality of a set of elements that normally generates the group. One obtains a lower bound by counting the number of factors in the abelianization, viewed as a direct sum. Providing other estimates from below has proved to be notoriously difficult. Such estimates are also of interest in 3-manifold topology, as the normal rank of $\pi_1(M)$ is a lower bound of the so-called Dehn surgery number of M^3; see Section 1.1.

A longstanding open problem posed by Wiegold in 1976 (Problem FP14 in [BMS02], Problem 5.52 in [KM23]), asks if there exist finitely generated perfect groups with normal rank greater than one, for which we give a positive answer:

Theorem A. *There exist finitely generated (even finitely presented) perfect groups that have normal rank greater than* 1.

There has been remarkably little progress on the problem since it was posed, besides Wiegold's elementary proof that finite perfect groups have normal rank 1. There are many candidates of such examples, such as free products of finitely generated torsion-free perfect groups. However, proving the lower bound on the normal rank remained intractable.

We prove the desired bound for free products of *left-orderable* groups, and thus provide a family of examples for Theorem A. A group G is left-orderable if it admits a total order $<$ such that $f < g$ implies that $hf < hg$ for all $f, g, h \in G$. For countable groups, this is equivalent to admitting an embedding in $\mathrm{Homeo}^+(\mathbb{R})$ [DNR14]. The landscape of left-orderable groups is vast and includes free and surface groups (more generally, locally

indicable groups [BH72]), braid groups, and virtually the fundamental groups of closed hyperbolic 3-manifolds [Ago13]. Finitely generated perfect left-orderable groups are plentiful with examples going back to the 1980s [GS87]. See [DNR14] for a comprehensive reference on left-orderable groups.

Here is the general result we prove, where $\langle\langle w \rangle\rangle$ is the smallest normal subgroup of G containing w.

Theorem B. *Any free product $G = A \star B$ of nontrivial left-orderable groups A and B has normal rank greater than* 1. *More precisely, for any $w \in G$ not conjugate into A, the natural map $A \hookrightarrow A \star B$ induces an* injection $A \hookrightarrow (A \star B)/\langle\langle w \rangle\rangle$.

We remark that the same estimate of normal rank holds for free products of groups, where each free factor admits a nontrivial left-orderable quotient.

Theorem A is a corollary of Theorem B by taking A and B to be finitely generated perfect left-orderable groups. Some prominent examples of finitely presentable perfect left-orderable groups include: the commutator subgroup of the braid group B_n for $n \geq 5$ [Deh00, DNR14], central extensions of Thompson's group T [GS87] and the $(2, 3, 7)$-triangle group [DNR14], and Higman's group [RT19]. The second author with Hyde constructed the first examples of finitely generated simple left-orderable groups [HL19], and even finitely presentable ones [HL25]. The former family contains continuum many groups (up to isomorphism).

1.1. **Connections to other problems and results.** We now explain how our results connect to other problems, and compare them with previously known results. The method of proof for Theorem B will be explained in Section 2.

Most known approaches towards the Wiegold problem involve showing that a free product $G = \star_{i \in I} A_i$ of nontrivial groups A_i has normal rank greater than 1, under suitable assumptions.

For $|I| \geq 3$, an unsolved conjecture of Cameron Gordon [Gor83, Conjecture 9.5] predicts that such a free product always has normal rank greater than 1. When the factors are finite cyclic groups, this is the Scott–Wiegold conjecture, confirmed by James Howie [How02]. It seems tempting to make this work for a free product of three (even finite) perfect groups, but no one has succeeded.

When $|I| = 2$, it is well-known that torsion can enable a free product $G = A \star B$ to have normal rank 1. For example, $G/\langle\langle w \rangle\rangle$ is trivial for $w = ab$ in $G = (\mathbb{Z}/2) \star (\mathbb{Z}/3)$, where a and b are generators of $\mathbb{Z}/2$ and $\mathbb{Z}/3$ respectively. The same argument works when A, B are simple groups containing torsion elements of coprime orders. So some assumptions should be imposed on A and B for G to have normal rank greater than 1.

Since the early 1980s, Theorem B was known to be true under the much stronger assumption that each free factor is *locally indicable*, meaning that every nontrivial finitely generated subgroup surjects onto $\mathbb{Z}$; see the independent work of Brodskiĭ [Bro84], Howie [How81] and Short [Sho83]. However, finitely generated locally indicable groups can never be perfect. Theorem B is also suspected to hold under the weaker assumption that each free factor is torsion-free; see Kirby's (1970s) problem list [Kir78, Problem 66] contributed

by Freedman. Such generalizations are challenging due to the lack of structures. Local indicability implies left-orderability, which in turn implies torsion-freeness.

If one takes $B = \mathbb{Z}$ in Theorem B, it becomes easier to show that G has normal rank greater than one, but there is a surjection $p : G = A \star \mathbb{Z} \to \mathbb{Z}$, and so G is not perfect. The Howie (or Kervaire–Laudenbach) conjecture [How81] asserts that Theorem B holds for $G = A \star \mathbb{Z}$ with any nontrivial group A whenever $p(w) \neq 0$. This has been confirmed when A is residually finite (by Gerstenhaber–Rothaus [GR62]), hyperlinear (by Pestov [Pes08]), or torsion-free[1] (by Klyachko [Kly93] and a recent new proof by the first author [Che25]). Actually, if A is torsion-free, the Levin conjecture [Lev62] asserts that the result holds with the weaker assumption that w is not conjugate into A. Theorem B confirms this for A left-orderable since $B = \mathbb{Z}$.

Considerable interest in the notion of normal rank comes from 3-manifold topology. The Lickorish–Wallace theorem [Wal60, Lic62] shows that any closed, orientable, connected 3-manifold M is the result of a Dehn surgery on some n-component link L in S^3, and the minimal number n, called the *Dehn surgery number*, has been considered as a complexity of M. There are many articles studying this [Auc97, HKL16, HL18, SZ22], but it is very difficult to give lower bounds. In fact, no example of Dehn surgery number greater than two is known; see the recent work [LP24] for a more detailed summary of the relevant work. The normal rank of $\pi_1(M)$ is a natural lower bound of the Dehn surgery number, since the fundamental group of $S^3 \setminus L$ has normal rank at most n, and so is the fundamental group of M if it is obtained from a Dehn surgery on $S^3 \setminus L$ since $\pi_1(M)$ is a quotient of $\pi_1(S^3 \setminus L)$. In this view, the Gordon conjecture [Gor83, Conjecture 9.5] is a generalization of the three-summand conjecture: A connected sum of three 3-manifolds not homeomorphic to S^3 must have Dehn surgery number greater than 1. This is in turn related to the cabling conjecture [GAnS86].

It is natural to ask if the following generalization of Theorem B holds.

Question 1.1. *Let G be a free product of n nontrivial left-orderable groups. Is the normal rank of G at least n?*

A positive answer would imply that the connected sum of n 3-manifolds with nontrivial left-orderable fundamental groups must have Dehn surgery number at least n. Left-orderability of fundamental groups of 3-manifolds is also well-studied in the context of the L-space conjecture [BGW13].

There have been some attempts to approach the Wiegold problem from other directions, which still remain mysterious. In the article [OT13], Osin and Thom provide a conjectural connection between the notion of normal rank of a countable torsion-free group and its first l^2-betti number, motivated by the Wiegold problem. In [MOT12], Monod, Ozawa and Thom formulate a version of the Wiegold problem for *irng*'s which are rings that are possibly non-unital, demonstrating that a solution in this setting would lead to a solution of the original Wiegold problem. Both approaches remain unsolved.

[1] under the stronger assumption $p(w) = \pm 1$

2. The structure of the proof

We sketch the proof of Theorem B and introduce the two main ingredients.

The notion of *right-orderability* is defined in the same way as left-orderability, except that the total order is required to be invariant under right multiplication instead. A group admits a left-order if and only if it admits a right-order. In this article, all actions will be right actions, so we shall use the notion of right-orderability in the rest of this paper.

Let A, B be right-orderable groups and $G = A \star B$. Up to conjugation, we express w as a cyclically reduced word:

$$w = a_1 b_1 \cdots a_n b_n \in G \qquad a_i \in A \setminus \{id\}, b_i \in B \setminus \{id\}.$$

Suppose some $a \in A \setminus \{id\}$ lies in the subgroup $\langle\langle w \rangle\rangle$ normally generated by w, we get an equation of the form:

$$a = (g_1 w^{n_1} g_1^{-1})(g_2 w^{n_2} g_2^{-1}) \cdots (g_k w^{n_k} g_k^{-1}),$$

where $k \in \mathbb{Z}_+$, each $g_i \in G$ and $n_i \in \mathbb{Z} \setminus \{0\}$ for all $1 \leq i \leq k$.

This gives rise to a surface map $f : S \to X$, where X is a $K(G, 1)$, S is a sphere with $k+1$ boundary components, and f takes the boundary components to loops in X representing the conjugacy classes of $a, w^{n_1}, \cdots, w^{n_k}$ respectively; see Figure 2. We aim to show that such a surface map is too simple to exist in the sense that the complexity of the surface, measured by $-\chi(S) = k - 1$, is strictly less than the complexity of the boundary, measured by $\sum_i |n_i|$ (which is at least k).

We show that the inequality should go the other way in Theorem C below, for a more general class of surfaces (allowing genus and so on), which we call *w-admissible surfaces*; see Definition 3.1. For such a surface S, we measure the complexity of the boundary by a notion of degree $\deg(S)$, which is an *unsigned* count of the number of copies of $w^{\pm 1}$ on the boundary. We ask the surface to be boundary-incompressible (Definition 3.3) to ensure a genuine count of $\deg(S)$. The surface S above arising from equations fits into the definition, where $\deg(S) = \sum_i |n_i|$, and it is boundary-incompressible if we take an equation of minimal length k; see Example 3.5. Thus the inequality we show in Theorem C below gives a contradiction.

Theorem C. *Let $G = A \star B$ be a free product of right-orderable groups A and B, and let w be an element of G not conjugate into A or B. Then for any boundary-incompressible w-admissible surface S without 2-sphere or disk components, we have*

$$-\chi(S) \geq \deg(S).$$

If we express w as a cyclically reduced word $a_1 b_1 \cdots a_n b_n$, Theorem C holds under the weaker assumption that the finitely generated subgroups $A' = \langle a_1, \ldots, a_n \rangle$ and $B' = \langle b_1, \ldots, b_n \rangle$ are right-orderable. Similarly for Theorem D below; see Theorem 4.1.

A similar result in the context of HNN extensions was recently shown by the first author [Che25, Theorem A] to give a new proof of the Klyachko theorem [Kly93]. These are inspired by the spectral gap property in the context of stable commutator length [DH91, Che18, CH19, Heu19], where the degree is the *signed* count $|\sum_i n_i|$. In that context,

the analog of Theorem C holds under the weaker torsion-freeness assumption on the free factors [Che18]. Here we are interested in the stronger inequalities involving the *unsigned* degree, for which we impose stronger assumptions. Similar unsigned or unoriented variants of stable commutator length has been studied by Duncan–Howie, Larsen Louder, Doron Puder, and Henry Wilton [DH91, LW24, Wil18, Wil24, PSEWS25], which have been useful in understanding surface subgroups, one-relator groups, and more.

To prove Theorem C, we have two main ingredients, one topological and the other dynamical. First, we use a topological argument that gives the desired estimate of the Euler characteristic (Theorem D below), assuming the existence of a combinatorial labeling. Such a labeling can be derived from a dynamical condition that we call *relative stacking* (Definition 2.1), generalizing the notion of stacking from [LW17].

Theorem D. *For any free product $G = A \star B$ and $w \in A \star B$ expressed as a cyclically reduced word. If there is a relative stacking of w, then for any boundary-incompressible w-admissible surface S without sphere or disk components, we have*

$$-\chi(S) \geq \deg(S).$$

Here is the key notion of relative stacking.

Definition 2.1 (Relative stacking)**.** Consider an action $\sigma : G = A \star B \to \mathrm{Homeo}^+(\mathbb{R})$, not necessarily faithful. For a cyclically reduced word $w = a_1 b_1 \cdots a_n b_n$ with $a_i \in A \setminus \{id\}, b_i \in B \setminus \{id\}$, the *trajectory* of some $x \in \mathbb{R}$ under w is the multiset of points:

$$\Omega(w, x) = \{x \cdot \sigma(a_1 b_1 \dots a_i b_i) \mid 1 \leq i \leq n\} \cup \{x \cdot \sigma(a_1 b_1 \dots a_i) \mid 1 \leq i \leq n\},$$

which are the images of x under the action of the nonempty prefixes of w.

We say the trajectory of x under w is *stable* if:

(1) Each element in $\Omega(w, x)$ occurs exactly once.
(2) $x \cdot \sigma(w) = x$.

A *relative stacking* of w is a right action σ of $A \star B$ on $\mathbb{R}$ by orientation-preserving homeomorphisms of the real line and a point $x \in \mathbb{R}$, such that $\Omega(w, x)$ is stable.

The second main input is to show the existence of relative stacking. It only works when w is not a proper power, but Theorem C is easier when w is a proper power. In fact, for the purposes of our main Theorem B, we only need to consider w which are not proper powers since $\langle\langle w^n \rangle\rangle \subset \langle\langle w \rangle\rangle$.

Theorem E. *Let $G = A \star B$ be a free product of countable right-orderable groups A and B. For any $w \in A \star B$ cyclically reduced of length at least 2, a relative stacking of w exists if and only if w is not a proper power.*

To prove Theorem E, we provide a method to solve certain equations and inequations over the group where the variables are a G-action on $\mathbb{R}$ and a chosen point in $\mathbb{R}$. This is done using systematic blow-ups of fixed group actions on $\mathbb{R}$, reducing it to the following problem: Given any nonempty proper prefix u of w, find an action $\sigma : A \star B \to \mathrm{Homeo}^+(\mathbb{R})$ and $x \in \mathbb{R}$ that satisfies $x \cdot \sigma(w) = x$ and $x \cdot \sigma(u) \neq x$. For this, we build an action σ with a closed interval I such that:

(1) $I \cdot \sigma(w) \subseteq I$.
(2) $I \cdot \sigma(u) \cap I = \emptyset$.

Such an action is built by developing a method that we call *dynamical arrangements*, which are careful combinatorial encodings of actions of $A \star B$ on $\mathbb{R}$. This finishes the proof since the first item ensures the existence of a point $x \in I$ fixed by $\sigma(w)$ using the intermediate value theorem, and the second item ensures the $\sigma(u)$ has no fixed point in I.

2.1. **Organization of the paper.** We give the topological backgrounds on w-admissible surfaces in Section 3 and then prove Theorem D in Section 4. Then we show the existence of relative stacking in Section 5. Finally we give the formal proofs of main results in Section 6.

3. Admissible surfaces

In this section, we define the necessary terminology about admissible surfaces and introduce the basic facts. Most of this is parallel to what is developed in the previous work of the first author [Che25], but here we focus on free products instead of HNN extensions. We include all details for completeness.

Given a group G and a collection of proper subgroups $\{A_i\}_{i\in I}$, where I could be empty, let X be a connected topological space with $\pi_1(X) = G$. The only case of interest in this paper is when $G = \star_{i\in I} A_i$ with $|I| = 2$.

Definition 3.1 (w-admissible)**.** Given $w \in G$, a map $f : S \to X$ from a compact oriented surface S is called a *w-admissible surface* in G (or X) relative to $\{A_i\}_{i\in I}$ if:

(1) The image of each boundary component of S
   - either represents the conjugacy class of w^n for some $n \in \mathbb{Z} \setminus \{0\}$ (we refer to the union of components of this kind as the *w-boundary*),
   - or represents a conjugacy class in G that intersects some A_i nontrivially (we refer to a boundary component of the this kind as an *A_i-boundary*).
(2) We require that the w-boundary is non-empty.

The A_i-boundary can be empty, but it is important to allow its existence (see Example 3.5). See the left of Figure 1 for an illustration.

We denote a (relative) w-admissible surface by the pair (f, S), but we will often simply denote it as S unless we would like to emphasize the map f.

For each w-boundary component representing w^n, we define its *degree* to be $|n|$. Then we define the *degree* of S as the sum of degrees over all w-boundary components and denote it as $\deg(S)$. For instance, the surface S in Figure 1 has degree $k + m + n$.

Remark 3.2. This is analogous to the definition of admissible surfaces relative to a collection of subgroups in the context of relative stable commutator length; see [Che20, Definition 2.8], and also [Cal09, Notation 2.5] for the absolute version. However, in that context, the degree is counted *with signs*, namely, a boundary component representing w^n with $n < 0$ contributes $-n$ to the degree. The unsigned version of degree here is an upper bound of the absolute value of the signed version.

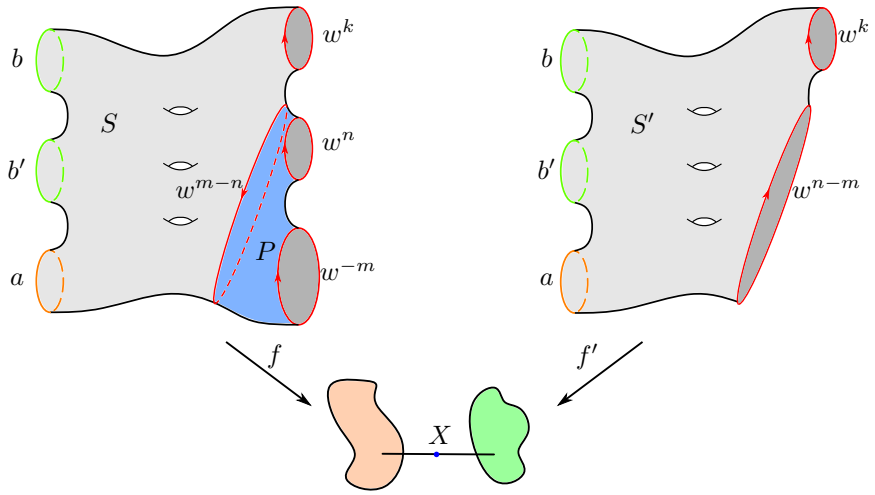

FIGURE 1. On the left is a w-admissible surface (f, S) into a space X with $\pi_1(X) = A \star B$, where S has two boundary components representing $b, b' \in B$, one boundary component representing $a \in A$, and three w-boundary components representing w^k, w^n, w^{-m} for some $k, m, n \in \mathbb{Z}_+$. The subsurface P witnesses its boundary-compressibility, and $S' = S \setminus P$ on the right is the simplified w-admissible surface, whose boundary representing w^{n-m} (with the orientation induced from S') needs to be further capped off by a disk if $m = n$.

To avoid a dummy count of degree, we will often require w-admissible surfaces to be *boundary-incompressible*, following [Che25, Definition 2.4].

Definition 3.3 (boundary-incompressibility)**.** A w-admissible surface (f, S) relative to $\{A_i\}_{i\in I}$ is *boundary-compressible* if there is an embedded subsurface $P \subset S$ homeomorphic to a pair of pants, such that two boundary components of P are on the w-boundary of S and represent the conjugacy class of w^n and w^{-m} for some $m, n \in \mathbb{Z}_+$, and the third boundary component of P is in the interior of S and, with the induced orientation from P, represents the conjugacy class of w^{m-n} under the map f; see the left of Figure 1.

We say (f, S) is *boundary-incompressible* if there is no such P.

Remark 3.4. One can compress a boundary-compressible w-admissible surface S as follows to get a simpler surface that is either w-admissible again or has no w-boundary left; see Figure 1 for an illustration. Let S' be the closure of $S \setminus P$ equipped with the restriction of f. This "merges" the two w-boundary components of S representing w^n and w^{-m} into one that represents the conjugacy class of w^{n-m} (with the induced orientation from S'). If $m \neq n$, then (f, S') is a w-admissible surface already. If $m = n$, then the new boundary is null-homotopic in X and thus bounds a disk $g : D \to X$. We use this to cap off S' to get a surface $S'' := S' \cup D$ with a well-defined map f'' using f and g, so that (f'', S'') is a w-admissible

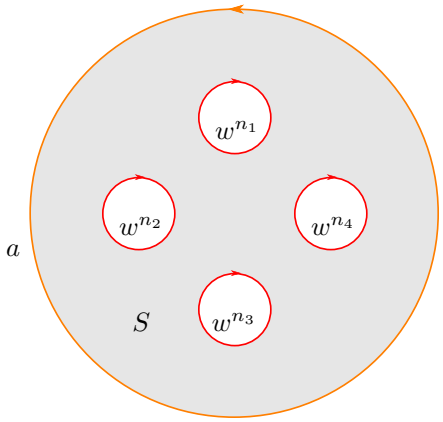

FIGURE 2. A w-admissible surface S corresponding to an equation of the form (3.1) with $k = 4$.

surface except that possibly there is no w-boundary left. The surfaces S' and S'' are simpler in the sense that $-\chi(S') = -\chi(S) - 1 < -\chi(S)$ and $-\chi(S'') = -\chi(S') - 1 < -\chi(S)$.

The pair of pants P in Definition 3.3 shrinks down to a neighborhood of the union of the two w-boundary components C, C' with a proper arc $\gamma \subset S$ connecting them. One can characterize boundary-incompressibility in terms of arcs (like γ); compare to the second part of [PSEWS25, Definition 3.1].

Here is the key example relating w-admissible surfaces to equations in a group.

Example 3.5. Let $a \neq id \in A$, which we treat as an element in $G = A \star B$. Let $w \in G$ be an element not conjugate into A or B. Let $\langle\langle w \rangle\rangle$ be the subgroup normally generated by w, then $a \in \langle\langle w \rangle\rangle$ if and only if one can express a in an equation of the form below:

$$a = (g_1 w^{n_1} g_1^{-1})(g_2 w^{n_2} g_2^{-1}) \cdots (g_k w^{n_k} g_k^{-1}), \tag{3.1}$$

where $k \in \mathbb{Z}_+$, each $g_i \in G$ and $n_i \neq 0 \in \mathbb{Z}$ for all $1 \leq i \leq k$.

Each such equation corresponds to a surface map $f : S \to X$, where S is a sphere with $k+1$ boundary components, where one boundary component represents the conjugacy class of a, and the remaining k components represent w^{n_i} for $1 \leq i \leq k$ respectively; see Figure 2. This makes (f, S) a w-admissible surface relative to $\{A, B\}$ (or just $\{A\}$), whose degree is $\sum_i |n_i|$.

If S is boundary-compressible, then the compression as in Remark 3.4 yields a w-admissible surface S' with fewer w-boundary components, which corresponds to an equation with strictly smaller k. Since a is nontrivial and the corresponding component in S does not change in the process, we cannot run out of w-boundary components. It follows that, if we consider an equation for a of the form (3.1) with minimal k, then the corresponding w-admissible surface S must be boundary-incompressible. These are the surfaces that we will apply our spectral gap Theorem C to.

We measure the complexity of a (w-admissible) surface S by the negative modified Euler characteristic $-\chi^-(S)$ defined as follows: For each connected component Σ of S, define $\chi^-(\Sigma) := \min(\chi(\Sigma), 0)$; then define $\chi^-(S)$ as the sum of $\chi^-(\Sigma)$ over all components of S. Equivalently, $\chi^-(S)$ is the Euler characteristic of S after removing all disk or sphere components. For our purposes, we can restrict our attention to w-admissible surfaces where each connected component contains some w-boundary component, so there is no disk (if w has infinite order) or sphere component, and hence the complexity is simply $-\chi^-(S) = -\chi(S)$.

3.1. **A normal form.** In this subsection, we focus on the case of $w \in G = A \star B$ and introduce a decomposition of any w-admissible surface into a *simple normal form*, after possibly simplifying the surface first. This allows us to put specific structures on the surface to aid our Euler characteristic estimate.

The (simple) normal form we define below is similar to the one used in [Che25], both should generalize to graphs of groups/spaces and are adapted from the (simple) normal form in [Che20] of admissible surfaces in the context of stable commutator length.

Since we will focus on the case of a free product $G = A \star B$ and $w \in G$ not conjugate into A or B, it is understood that the w-admissible surfaces we consider are *relative to* $\{A, B\}$.

Let X_A and X_B be pointed $K(A,1)$ and $K(B,1)$ spaces respectively. Let $X = X_A \sqcup [-1,1] \sqcup X_B/\sim$, where the equivalence relation $\sim$ glues -1 (resp. 1) to the base point of X_A (resp. X_B). Then X is a $K(G,1)$ space, which contains a point $\star$ corresponding to the midpoint 0, and we treat it as the base point of X. Let $\widehat{X}_A := X_A \cup [-1,0] \subset X$, which is a copy of $K(A,1)$ based at $\star$, and similarly let $\widehat{X}_B := X_B \cup [0,1] \subset X$; see Figure 3.

Fix any element $w \in G$ that is not conjugate into A or B. Up to conjugation, we assume that w is written as a cyclically reduced word $w = a_1 b_1 \cdots a_n b_n$ for some $n \in \mathbb{Z}_+$, where each $a_i \neq id \in A$ and $b_i \neq id \in B$. Denote the *word length* of w as $|w| := 2n$. Note that any such w has infinite order.

We can represent w as a map $f_w : S^1_w \to X$ from an oriented circle S^1_w, so that

$$J_w := f_w^{-1}(\star) \subset S^1_w$$

consists of $|w|$ points, which we refer to as the *junctures*. The junctures cut S^1_w into $|w|$ arcs $\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n$ in cyclic order, where each α_i (resp. β_i) is mapped by f_w to a loop in X supported on $\widehat{X}_A$ (resp. $\widehat{X}_B$) based at $\star$ representing $a_i \in A$ (resp. $b_i \in B$); see Figure 3.

For any w-admissible surface (f, S), up to homotopy, we assume that the restriction of f to each w-boundary component C factors as $C \cong S^1 \xrightarrow{p} S^1_w \xrightarrow{f_w} X$, where p is a covering map whose degree d is the degree of the component C. Then p pulls back the set of junctures J_w to a set $J_C := p^{-1}(J_w)$ of $d|w|$ points on C, which we also refer to as junctures; see Figure 3. These junctures cut C into $d|w|$ *segments*, each is mapped to some $\alpha_i^{\pm 1}$ or $\beta_i^{\pm 1}$ under the map p, where the sign is $+1$ if and only if C represents a positive power of w.

We may also assume up to a homotopy that each A-boundary or B-boundary of S has image under f away from $\star$.

Since $\star$ is locally a submanifold of codimension one in X, and $f|_{\partial S}$ is already transverse to $\star$, we may assume that f is transverse to $\star$, up to a homotopy of f rel ∂S. Then by

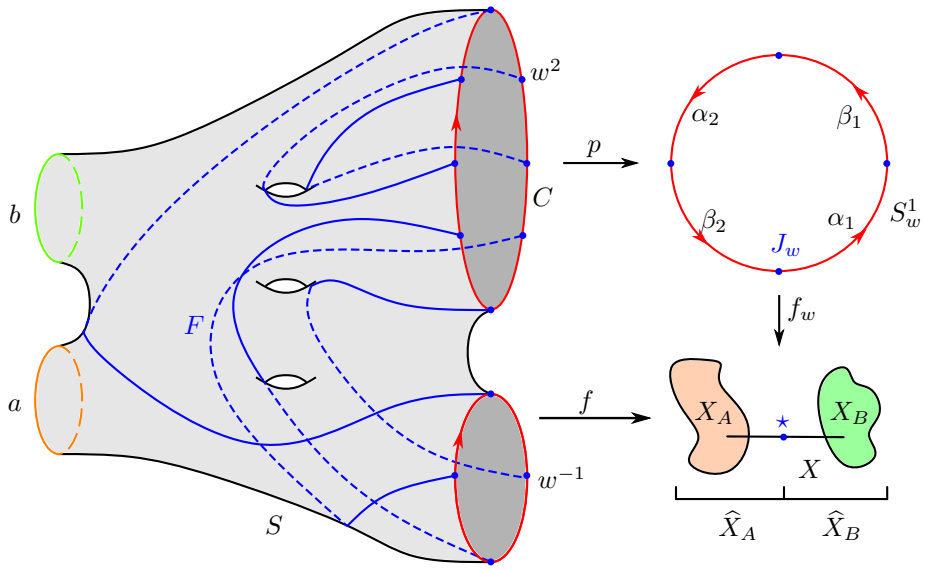

FIGURE 3. The set $J_w = f_w^{-1}(\star)$ on the circle S^1_w. The set of disjoint proper arcs $F = f^{-1}(X_C)$ is a set of embedded disjoint proper arcs in a reduced w-admissible surface S. The w-boundary component C has a cover map p to S^1_w that pulls back J_w to J_C.

transversality, $F := f^{-1}(\star)$ is a properly embedded 1-submanifold of S. That is, F is a finite disjoint union of embedded proper arcs and embedded loops.

Definition 3.6 (reduced)**.** We say a w-admissible surface S is *reduced* if each connected component of S intersects the w-boundary, and $F = f^{-1}(\star)$ contains no loop, that is, F is a finite disjoint union of embedded proper arcs; see Figure 3. In particular, there are no closed or disk components since w has infinite order, so $\chi^-(S) = \chi(S)$ if S is reduced.

We can always simplify S to a reduced one by the next lemma.

Lemma 3.7. *If (f, S) is a w-admissible surface in X, then there is a w-admissible surface (f', S') with $\deg(S') = \deg(S)$ and $-\chi^-(S') \le -\chi^-(S)$ such that S' is reduced. Moreover, (f', S') is boundary-incompressible if (f, S) is.*

Proof. First we discard components of S that are disjoint from the w-boundary. This preserves $\deg(S)$ and boundary-incompressibility, and it does not increase $-\chi^-(S)$.

Now assume each component of S witnesses the w-boundary. Then S has no closed or disk components as w has infinite order, so $\chi^-(S) = \chi(S)$.

Suppose there is a loop $L \subset F$, which is embedded in S. Let $c : D \to X$ be the constant map taking a closed disk D to $\star \in X$. Cutting S along L creates two new boundary components, and we close them up by gluing in two copies of D along the boundary. Denote the new surface by S'. Note that $f|_L$ agrees with $c|_{\partial D}$, so we obtain a well-defined map $f' : S' \to X$.

Then (f', S') is w-admissible with $\deg(S') = \deg(S)$ since boundary components of S' exactly correspond to those of S, on which the map f' agrees with f.

Denote by Σ the component of S containing L, which becomes $\Sigma' \subset S'$, which is either one or two components of S' depending on whether L separates Σ. Note that $-\chi(\Sigma') = -\chi(\Sigma) - 2$ by Mayer–Vietoris. By our assumption above, Σ is not closed, so L cannot bound disks on both sides.

(1) If L bounds a disk on one side, the process above creates a sphere component, and the other component is homeomorphic to Σ. In this case we delete the sphere component and redefine S' to be the remaining surface (Figure 4), in which case $\chi^-(\Sigma') = \chi(\Sigma') = \chi(\Sigma) = \chi^-(\Sigma)$.
(2) Otherwise, this does not create any sphere component and can create at most two disk components[2], so $\chi^-(\Sigma') \geq \chi(\Sigma') - 2$. Thus

$$-\chi^-(\Sigma') \leq -\chi(\Sigma') + 2 = -\chi(\Sigma) = -\chi^-(\Sigma).$$

Hence in any case we have $-\chi^-(S') \leq -\chi^-(S)$. For each component of Σ', homotope f' slightly to push $f'(D)$ away from $\star$ to eliminate the loop L from $F' = f'^{-1}(\star)$ so that F' has fewer loops than F.

It remains to see that (f', S') inherits the boundary-incompressibility of (f, S). Suppose (f', S') is boundary-compressible due to an embedded pair of pants P. Denote the boundary components of P as L_1, L_2 and L_3, where L_1 and L_2 are w-boundary components of S' (coming from S) and L_3 is in the interior of S'. Up to an isotopy, we may shrink P to a tubular neighborhood of $L_1 \cup L_2 \cup \gamma$ for some proper embedded arc γ connecting L_1 and L_2. We can homotope γ away from the new disks in the construction of S', which does not change the homotopy class of $f'(\partial L_3)$; see Figure 4. So we may assume that P is disjoint from the new disks and thus correspond to a pair of pants in S that witnesses the boundary-compressibility of (f, S).

The process above strictly decreases the number of loop components in F, so by repeating the whole process above finitely many times we arrive at a desired reduced (f', S'). □

Now assume that (f, S) is already a reduced w-admissible surface. In particular, $F = f^{-1}(\star)$ is a finite disjoint union of proper arcs, whose endpoints are necessarily junctures on the w-boundary of S. Note that any juncture is contained in exactly one arc in F by disjointness. Then F cuts S into two closed (as subsets) possibly disconnected subsurfaces $S_A := f^{-1}(\widehat{X}_A)$ and $S_B := f^{-1}(\widehat{X}_B)$ so that $S = S_A \cup_F S_B$; see the left of Figure 5.

There are two possible kinds of boundary components of S_A:

(1) A-boundary components, exactly corresponding to the A-boundary components of S (which are disjoint from F); see the orange loops on the right of Figure 5; and
(2) *polygonal boundary components*, each of which has an even number of sides, alternating between segments on some w-boundary of S and proper arcs in F, the latter are called *turns*; see the red and blue arcs on the right of Figure 5.

[2]Actually, it is also easy to show that this does not occur.

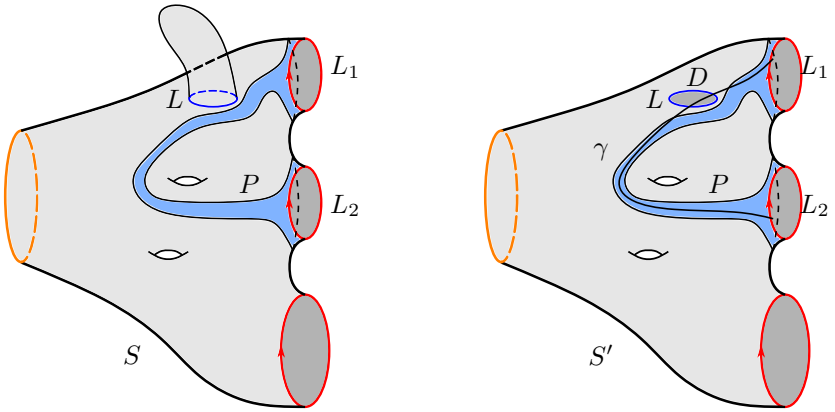

FIGURE 4. Compress a w-admissible surface S along a loop $L \subset F$ to obtain S'. Push the proper arc γ away from the disk D bounding L in S' to isotope the pair of pants $P \simeq L_1 \cup L_2 \cup \gamma$ so that it "lifts" to S.

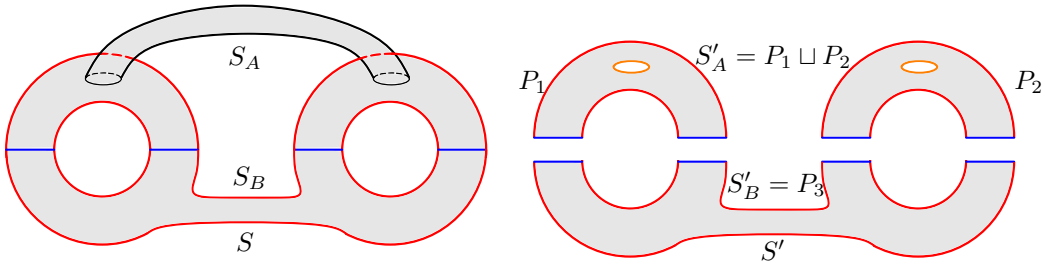

FIGURE 5. Left: The proper arcs in F cut a reduced w-admissible surface S into S_A and S_B, and in this example S_A has two polygonal boundary components and S_B has one. Right: Simplifying S as in Lemma 3.9 gives a w-admissible S' in simple normal form, which is a union of a disk-piece P_3 and two annulus-pieces P_1 and P_2, where $d(P_1) = d(P_2) = 2$, and $d(P_3) = 4$.

Since $f(S_A) \subset \widehat{X}_A$, each polygonal boundary component of S_A is mapped to a loop in $\widehat{X}_A$, representing some conjugacy class in A, referred to as the *winding class* of this polygonal boundary component. Note that the winding class is necessarily trivial if the polygonal boundary component bounds a disk component of S_A.

There is a similar structure on S_B and we define the terminology similarly. The surface S is obtained by gluing S_A and S_B along the turns.

Definition 3.8 ((simple) normal form, disk-pieces, and annuli-pieces)**.** We call each connected component of S_A or S_B a *piece*. We refer to such a decomposition of S into pieces as the *normal form* of the reduced w-admissible surface (f, S). We say the normal form is *simple* if each piece has exactly one polygonal boundary component and is homeomorphic to either a disk or an annulus; see the right of Figure 5.

In the simple normal form, we refer to the two kinds of pieces as *disk-pieces* and *annulus-pieces* depending on the topological type. For each such piece P, define its *valence* as $d(P) := k$ if the unique polygonal boundary component has $2k$-sides (i.e. k segments), where $k \in \mathbb{Z}_+$. Denote the Euler characteristic of such a piece by $\chi(P)$, which is 1 (resp. 0) if P is a disk-piece (resp. annulus-piece).

We can always simplify a w-admissible surface into one that admits a simple normal form. In particular, a piece in S_A with more than one A-boundary component can be simplified.

Lemma 3.9. *For any w-admissible surface (f, S), there is another w-admissible surface (f', S'), such that (f', S') has a simple normal form, with $-\chi^-(S') \leq -\chi^-(S)$ and $\deg(S') = \deg(S)$. Moreover, (f', S') is boundary-incompressible if (f, S) is.*

Proof. By Lemma 3.7, we may assume that (f, S) is reduced. Decompose S into the normal form by the process above. Then each piece must contain at least one polygonal boundary component, since otherwise this piece must be a component of S disjoint from the w-boundary.

In particular, any piece homeomorphic to a disk must have a unique polygonal boundary component, thus it is a disk-piece.

Consider any piece P that is not a disk. Then $\chi(P) \leq 0$. Let us assume that $P \subset S_A$, and the other case is similar. For each polygonal boundary component, cut out a tubular neighborhood N of it. Then N is an annulus and we treat the new boundary component of N as an A-boundary as it is mapped to a loop in $\widehat{X}_A$. This cuts P into some P' homeomorphic to P together with finitely many annuli like N. So $\chi(P') = \chi(P) \leq 0$, and throwing P' away does not increase $-\chi^-(S)$. Moreover, the operation above is performed away from the polygonal boundaries, so it does not affect the gluing of S_A with S_B and does not change the w-boundary; see Figure 5.

Applying this to each such piece P in S, denote the new surface as S' and define f' as the restriction of f. This makes (f', S') a reduced w-admissible surface with $-\chi^-(S') \leq -\chi^-(S)$ and $\deg(S') = \deg(S)$. Moreover, the normal form of S' consists of disk-pieces and annuli-pieces by construction. Since we are just deleting a subsurface of S to obtain S', boundary-incompressibility of S implies that of S'. □

Finally, we give a formula to compute the Euler characteristic of a w-admissible surface S in a simple normal form, which we will use in the next section for our estimates.

Lemma 3.10. *For any w-admissible surface S in a simple normal form, we have*

$$-\chi^-(S) = -\chi(S) = \sum_P \left[\frac{d(P)}{2} - \chi(P) \right],$$

where the summation is taken over all pieces P in the decomposition of S.

Proof. Any S in simple normal form is reduced. Hence S has no closed or disk components (as w has infinite order). Thus $\chi^-(S) = \chi(S)$.

Now by Mayer–Vietoris, using the fact that S_A and S_B are disjoint unions of pieces P, we have

$$\chi(S) = \chi(S_A) + \chi(S_B) - \chi(F) = \sum_P \chi(P) - |F|,$$

where $|F|$ denotes the number of proper arcs in F (each having Euler characteristic 1).

Note that each piece witnesses $d(P)$ proper arcs in F, namely the turns in P, and each proper arc in F is counted exactly twice, by a piece in S_A and one in S_B. Thus $|F| = \sum_P d(P)/2$. Combining this with the formula above, the desired equation follows. □

4. The spectral gap property via stackings

The purpose of this section is to prove Theorem D, which we restate as the following stronger version, allowing A and B to be uncountable. This is not essential, since a w-admissible surface in simple normal form only sees the countable subgroups A' and B'. So we will assume A and B to be countable in the proof below.

Theorem 4.1. *For any free product $G = A \star B$ and a cyclically reduced word $w = a_1 b_1 \dots a_n b_n \in A \star B$ with $n \in \mathbb{Z}_+$, $a_i \in A \setminus \{id\}$ and $b_i \in B \setminus \{id\}$. If there is a relative stacking of w in $G' = A' \star B'$, where A' (resp. B') is the subgroup generated by $a_1, \dots, a_n$ (resp. $b_1, \dots, b_n$), then for any boundary-incompressible w-admissible surface S in G, we have*

$$-\chi^-(S) \geq \deg(S).$$

We use the setup as in Section 3.1. By Lemma 3.9, it suffices to prove the theorem for any (f, S) that is in simple normal form. For our estimate of Euler characteristic, we need to put some additional structure on the pieces using the relative stacking.

A *prefix* u of w is a subword of w of the form $u = a_1 b_1 \dots a_i$ or $u = a_1 b_1 \dots a_i b_i$ for for some $1 \leq i \leq n$. There are exactly $|w| = 2n$ of them.

As in Section 3.1, we view w as a map $f_w : S^1_w \to X$ so that the set $J_w = f_w^{-1}(\star)$ of junctures divides the oriented circle S^1_w into $|w| = 2n$ oriented arcs $\alpha_1, \beta_1, \dots, \alpha_n, \beta_n$ in cyclic order. The map f_w takes each arc α_i (resp. β_i) to a loop based at $\star$ representing $a_i \in A$ (resp. $b_i \in B$). Note that the junctures correspond to prefixes u of w, by thinking of a juncture as where the prefix u ends, where the juncture between β_n and α_1 corresponds to the full word w.

Recall that a relative stacking of w gives a right action of G on $\mathbb{R}$ and a special point x whose trajectory $\Omega(w, x)$ is stable (i.e. this is a multiset without repeated elements and also $x \cdot w = x$), where each point in the trajectory is $x \cdot u$ for a nonempty prefix u of w. Hence a relative stacking gives an assignment $\lambda : J_w \to \mathbb{R}$, where the value of the juncture corresponding to a prefix u is $x \cdot u$. We record the data of a relative stacking into properties of this assignment λ as follows:

Lemma 4.2. *Given a relative stacking of w, the assignment $\lambda : J_w \to \mathbb{R}$ is* injective*, with the property that, for each arc α_i (resp. β_i) on S^1_w going from a juncture j to another juncture j', we have $\lambda(j) \cdot a_i = \lambda(j')$ (resp. $\lambda(j) \cdot b_i = \lambda(j')$), for the action given by the relative stacking.*

Proof. The injectivity of λ follows from the fact that the elements in $\Omega(w,x)$ are distinct. The other property is by definition except for the arc α_1: It goes from the juncture j corresponding to w to the juncture j' corresponding to the prefix a_1, but note that $\lambda(j) = x \cdot w = x$, so $\lambda(j) \cdot a_1 = x \cdot a_1 = \lambda(j')$ as desired. □

The properties of λ described above involve the right action from the relative stacking, and essentially encapsulate the stability of $\Omega(w,x)$. In what follows, we will just work with the assignment λ instead of the relative stacking.

In this form, interpreted topologically, this is analogous to the notion of stacking in [LW17]. In fact, one can generalize our notion of relative stacking to the setting of graphs of groups with trivial edge groups. If the vertex groups are also trivial, then it is equivalent to stacking in the sense of [LW17]. Our definition is inspired by but different from the notion of relative stacking in [Mil21].

Recall from Section 3.1 that the restriction of the map f to each w-boundary component C of S of some degree $d \geq 1$ factors as a map of the form $C \cong S^1 \xrightarrow{p} S^1_w \xrightarrow{f_w} X$ for a covering map p of degree d. The map p pulls back the junctures J_w on S^1_w to junctures $J_C := p^{-1}(J_w)$ on C, which cut C into $d|w|$ segments, each labeled by some a_i^s or b_i^s, where $s = \pm 1$ records whether p is orientation-preserving when the C is equipped with the orientation induced from S.

The map p pulls back the assignment λ to an assignment $\hat{\lambda}_C : J_C \to \mathbb{R}$ defined as $\hat{\lambda}_C := \lambda \circ p$, which is $|w|$-periodic when we go around J_C in the cyclic order. Define J_S as the disjoint union of J_C over all w-boundary components C, and define $\hat{\lambda} : J_S \to \mathbb{R}$ component-wise using $\hat{\lambda}_C$. Note that $|J_S| = |w| \deg(S)$.

As (f,S) is in simple normal form, it is reduced and $F = f^{-1}(\star)$ is a finite disjoint union of embedded proper arcs, which correspond to turns on the pieces. Each such arc γ connects two junctures $j, j' \in J_S$, and we *orient* γ so that it points from j to j' if $\hat{\lambda}(j) > \hat{\lambda}(j')$ and the other way around if $\hat{\lambda}(j) < \hat{\lambda}(j')$. The basic but important observation below shows that this defines an orientation on each arc $\gamma \subset F$ whenever (f,S) is boundary-incompressible.

Lemma 4.3. *If (f,S) is boundary-incompressible and in simple normal form, then for any arc $\gamma \subset F$ connecting junctures $j, j' \in J_S$, we have $\hat{\lambda}(j) \neq \hat{\lambda}(j')$.*

Proof. Let C and C' be the w-boundary components of S that j and j' belong to, respectively. Suppose $\hat{\lambda}(j) = \hat{\lambda}(j')$. Since the map λ is injective by Lemma 4.2, the junctures j and j' must correspond to the same juncture $j_* \in J_w$ on S^1_w. Assume that j_* is the one between α_i and β_i for some $1 \leq i \leq n$. The case where j_* is between some β_i and α_{i+1} is similar.

With the induced orientation from S, there is a segment s_1 on C (resp. s_1' on C') ending at j (resp. j') and a segment s_2 on C (resp. s_2' on C') starting at j (resp. j'); see the left of Figure 6. Suppose C (resp. C') represents the conjugacy class of w^n for some $n \neq 0 \in \mathbb{Z}$ (resp. $w^{n'}$ for some $n' \neq 0 \in \mathbb{Z}$). There are two cases for the labels on s_1 and s_2 depending on the sign of n, and similarly on the C' side:

(1) If $n > 0$ (resp. $n' > 0$), then s_1 (resp. s_1') is labeled by a_i and s_2 (resp. s_2') by b_i;

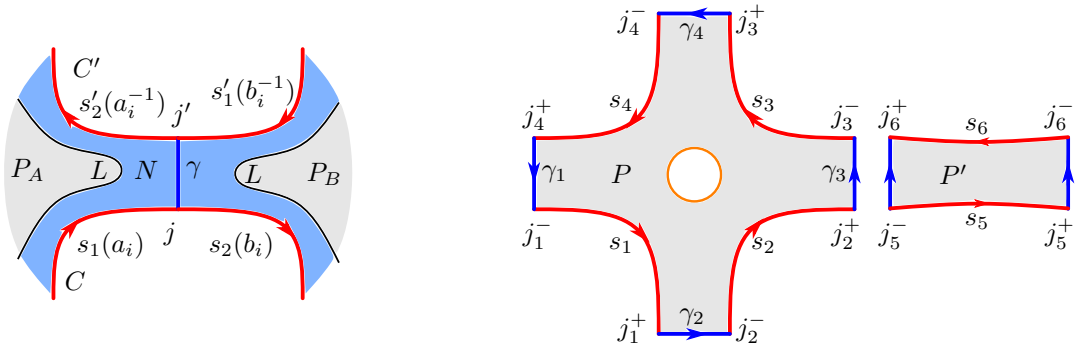

FIGURE 6. Left: The local structure of an arc $\gamma \in F$ connecting two junctures $j \in J_C$ and $j' \in J_{C'}$ that correspond to the same $j_* \in J_w$. A neighborhood N of $C \cup C \cup \gamma$ witnesses boundary-compressibility. Right: Two pieces P and P' in the simple normal form of some S, marked with $\hat{\lambda}$-orientations. In this example, $\mathrm{sc}(P) = 0$ and $\mathrm{sc}(P') = 2$.

(2) If $n < 0$ (resp. $n' < 0$), then s_1 (resp. s_1') is labeled by b_i^{-1} and s_2 (resp. s_2') by a_i^{-1}.

We now observe that n and n' must have opposite signs, and in particular $C \neq C'$. In fact, recall that F cuts S into two subsurfaces S_A and S_B, each of which consists of pieces. The arc γ appears as turns in exactly two pieces $P_A \subset S_A$ and $P_B \subset S_B$. Up to swapping j and j', let us assume that the turn on P_A with the induced orientation from P_A is from j to j'; see the left of Figure 6. Then s_1 and s_2' lie in P_A and must be labeled by something in the group A, so we must have $n > 0$ and $n' < 0$ based on the possibilities listed above.

Let N be a tubular neighborhood of $C \cup \gamma \cup C'$, which is homeomorphic to a pair of pants with boundary components C, C' and L for some embedded loop L in S. Note that f maps γ to $\star$, so it follows that L with the induced orientation from N represents the conjugacy class of $w^{-(n'+n)}$, which implies that (f, S) is boundary-compressible; see Definition 3.3 and take $m = -n'$. This contradicts our assumption and thus we cannot have $\hat{\lambda}(j) = \hat{\lambda}(j')$. □

With the lemma above, this defines an orientation on each arc in F if S is boundary-incompressible. Refer to this as the $\hat{\lambda}$-*orientation*. Based on this, we make the following definition:

Definition 4.4. For each segment s on the w-boundary of S connecting two junctures j_1 and j_2, let γ_1 and γ_2 be the arcs in F that contain j_1 and j_2 respectively. There are four cases for the $\hat{\lambda}$-orientations on γ_1 and γ_2:

(1) γ_1 points towards j_1, and γ_2 points towards j_2;
(2) γ_1 points away from j_1, and γ_2 points away from j_2;
(3) γ_1 points towards j_1, and γ_2 points away from j_2; and
(4) γ_1 points away from j_1, and γ_2 points towards j_2.

We say s is *consistent* if we have case (1) or (2), and we say s is *inconsistent* otherwise. For example, in the right part of Figure 6, the segment s_i is inconsistent for $1 \leq i \leq 4$ and is consistent for $i = 5, 6$.

The lemma below is a key observation that helps us estimate the number of (in)consistent segments on the w-boundary of S: Each copy of w (or w^{-1}) on ∂S witnesses at least two inconsistent arrows.

Lemma 4.5. *For any boundary-incompressible w-admissible surface S in simple normal form, the total number of inconsistent segments is at least $2\deg(S)$, thus the total number of consistent segments is no more than $(|w| - 2)\deg(S)$.*

Proof. It suffices to prove the lower bound on the total number of inconsistent segments, as the total number of segments is $|w|\deg(S)$.

Since $\lambda : J_w \to \mathbb{R}$ is injective by Lemma 4.2, there is a unique juncture $j_M \in J_w$ (resp. $j_m \in J_w$) for which $\lambda(j_M) \geq \lambda(j)$ (resp. $\lambda(j_m) \leq \lambda(j)$) for all $j \in J_w$. Clearly $j_M \neq j_m$.

For each w-boundary component C of degree d, there are exactly d junctures in J_C corresponding to j_M (resp. j_m), which we call *maximal* (resp. *minimal*). Moreover, the d maximal and d minimal junctures sit in an alternating way on C.

For each maximal (resp. minimal) juncture j, the adjacent arc $\gamma \subset F$ must have $\hat{\lambda}$-orientation pointing away from (resp. towards) j. Thus between each adjacent pair of maximal and minimal junctures, there is at least one segment that is inconsistent. Thus we must have at least $2d$ inconsistent segments on each w-boundary component of degree d. Taking the sum over all w-boundary components completes the proof. □

Another key observation (Lemma 4.6) is to count the number of consistent segments in each piece P. Recall that the valence $d(P)$ of a (disk- or annulus-) piece P is the number of segments on the unique polygonal boundary component of P. Among these $d(P)$ segments, denote the number of consistent segments as $\mathrm{sc}(P)$, the number of *sign changes* in P.

The proof of Lemma 4.6 below justifies the terminology. This is where we make use of the action from the relative stacking.

Lemma 4.6. *For any (disk- or annulus-) piece P, the number of sign changes $\mathrm{sc}(P) \in \mathbb{Z}_{\geq 0}$ is even. Moreover, $\mathrm{sc}(P) > 0$ for each disk-piece P.*

Proof. Equip the polygonal boundary of P with the orientation induced from P. Recall that the polygonal boundary of P alternates with turns and segments, so we can enumerate them in the cyclic order as $\gamma_1, s_1, \ldots, \gamma_k, s_k$, where $k = d(P) \in \mathbb{Z}_+$ is the valence of P, each $\gamma_i \subset F$ is a turn and each s_i is a segment on some w-boundary component of S. Denote the endpoints of each s_i as j_i^- and j_i^+, which are junctures on some w-boundary, so that s_i goes from j_i^- to j_i^+. Then each γ_i with the topological orientation (induced from P) goes from j_{i-1}^+ to j_i^-, where indices are taken mod k and similarly below; see the piece P in the right part of Figure 6.

The topological orientation on each γ_i may or may not agree with the $\hat{\lambda}$-orientation defined earlier. Assign a number $\mathrm{sign}_i \in \{1, -1\}$ for each $1 \leq i \leq k$ so that $\mathrm{sign}_i = 1$ if and only if these two orientations on γ_i agree.

For each segment s_i, the topological orientation on γ_i points towards j_i^-, and the one of γ_{i+1} points away from j_i^+. Thus s_i is consistent if and only if $\mathrm{sign}_{i-1} = -\mathrm{sign}_i$.

Thus the number $\mathrm{sc}(P)$ of consistent segments is equal to the number of $i \in \{1, \dots, k\}$ with $\mathrm{sign}_{i-1} = -\mathrm{sign}_i$; see the pieces P and P' on the right of Figure 6 as examples. This is why we refer to it as the number of sign changes in P. Then clearly $\mathrm{sc}(P)$ is even.

Next we show $\mathrm{sc}(P) \neq 0$ if P is a disk-piece. To simplify the notation in the proof below, let us assume that P lies in S_A, and the other case works by symmetry. Then each segment s_i is labeled by some element $\varphi_i \in A$, where $\varphi_i = a_t^{\pm 1}$ for a letter a_t in the cyclically reduced expression of w, where $1 \leq t \leq n$. By Lemma 4.2 and the definition of $\hat{\lambda}$, we know

$$\hat{\lambda}(j_i^-) \cdot \varphi_i = \hat{\lambda}(j_i^+).$$

Moreover, since P is a disk-piece, the winding class is trivial, so

$$\varphi_1 \varphi_2 \cdots \varphi_k = id.$$

Suppose $\mathrm{sc}(P) = 0$. Then we have either $\forall i, \mathrm{sign}_i \equiv 1$ or $\forall i, \mathrm{sign}_i \equiv -1$. Assume without loss of generality that $\mathrm{sign}_i \equiv 1$ for all i, as the other case is similar; see the piece P (assuming it is a disk) on the right of Figure 6 as an illustration. That is, for each i, the $\hat{\lambda}$-orientation of each γ_i goes from j_{i-1}^+ to j_i^-, meaning

$$\hat{\lambda}(j_{i-1}^+) > \hat{\lambda}(j_i^-).$$

Then we have $\hat{\lambda}(j_k^+) > \hat{\lambda}(j_1^-)$ by taking $i = 1$. As φ_1 acts by an orientation-preserving homeomorphism of $\mathbb{R}$, we have

$$\hat{\lambda}(j_k^+) \cdot \varphi_1 > \hat{\lambda}(j_1^-) \cdot \varphi_1 = \hat{\lambda}(j_1^+) > \hat{\lambda}(j_2^-).$$

Then applying φ_2 to the inequality $\hat{\lambda}(j_k^+) \cdot \varphi_1 > \hat{\lambda}(j_2^-)$, we similarly obtain

$$\hat{\lambda}(j_k^+) \cdot \varphi_1 \varphi_2 > \hat{\lambda}(j_2^-) \cdot \varphi_2 = \hat{\lambda}(j_2^+) > \hat{\lambda}(j_3^-).$$

Inductively, this shows that, for all $1 \leq \ell \leq k$, we have

$$\hat{\lambda}(j_k^+) \cdot \varphi_1 \cdots \varphi_\ell > \hat{\lambda}(j_\ell^+) > \hat{\lambda}(j_{\ell+1}^-).$$

Taking the first inequality with $\ell = k$ and using $\varphi_1 \varphi_2 \cdots \varphi_k = id$, we observe that

$$\hat{\lambda}(j_k^+) = \hat{\lambda}(j_k^+) \cdot \varphi_1 \cdots \varphi_k > \hat{\lambda}(j_k^+),$$

which gives a contradiction. □

Corollary 4.7. *For any disk- or annulus-piece P, we have*

$$\mathrm{sc}(P) \geq 2\chi(P).$$

Proof. If P is an annulus-piece, then $2\chi(P) = 0 \leq \mathrm{sc}(P)$. If P is a disk-piece, then $\chi(P) = 1$, and $\mathrm{sc}(P) \geq 2$ by Lemma 4.6. □

Now we are in a place to prove Theorem 4.1.

Proof of Theorem 4.1. By Lemma 3.9, we may assume that S is in simple normal form. In particular, $\chi^-(S) = \chi(S)$. Using the notions above, note that for each piece P, the valence $d(P)$ counts the total number of all segments on its polygonal boundary, among which $\mathrm{sc}(P)$ of them are consistent segments. We know the sum of $d(P)$ over all pieces P in S is $|w|\deg(S)$, and the sum of $\mathrm{sc}(P)$ is at most $(|w|-2)\deg(S)$ by Lemma 4.5. Combining this with Lemma 3.10 and Corollary 4.7, we obtain the following estimate of $-\chi(S)$:

$$\begin{aligned} -\chi(S) &= \sum_P \left[\frac{d(P)}{2} - \chi(P)\right] \\ &\geq \frac{1}{2}\sum_P [d(P) - \mathrm{sc}(P)] \\ &\geq \frac{1}{2}[|w|\deg(S) - (|w|-2)\deg(S)] \\ &= \deg(S). \end{aligned}$$

□

The proof presented here is similar in spirit to the one in [DH91], but the novelty of our approach is the usage of $\hat{\lambda}$ as our labels of junctures, and its construction using actions on the line as presented in the next section. Besides, one can also phrase the topological argument above in terms of the LP duality method in [Che25, Section 4], where the cost function here is defined using the $\hat{\lambda}$-orientation. It can also be phrased using angle structures similar to the method used in [Mar24].

5. The existence of relative stackings

All actions in this section will be right actions, and for homeomorphisms $\lambda_1 : X \to Y$ and $\lambda_2 : Y \to Z$, the composition will be denoted as $\lambda_1\lambda_2$, rather than $\lambda_2 \circ \lambda_1$. A classical result states that a countable group G admits a faithful action by orientation-preserving homeomorphisms on the real line if and only if it is right-orderable. Given a right-order on G, one may construct such an action on $\mathbb{R}$ by first constructing a suitable order-embedding $G \to \mathbb{R}$ so that the natural right action on the image extends to an action by orientation-homeomorphisms on $\mathbb{R}$ [DNR14]. This is the so called *dynamical realization* of the order. Conversely, given an embedding $G \to \mathrm{Homeo}^+(\mathbb{R})$, one can extract a right-invariant order on G as follows. Choose an enumeration $(x_n)_{n\in\mathbb{N}}$ of $\mathbb{Q}$. Then $f < g$ if and only if for the smallest $n \in \mathbb{N}$ such that $x_n \cdot f \neq x_n \cdot g$, it holds that $x_n \cdot f < x_n \cdot g$. This is clearly a right invariant total order.

Consider two countable right-orderable groups A and B. Denote $G = A \star B$. Consider an action $\sigma : G \to \mathrm{Homeo}^+(\mathbb{R})$ (not necessarily faithful), and a word $w = a_1b_1 \dots a_nb_n$ such that $a_i \in A \setminus \{id\}, b_i \in B \setminus \{id\}$. Recall that a word w_1 is a *prefix* of w if it is a nonempty word of the form $a_1b_1 \dots a_i$ or $a_1b_1 \dots a_ib_i$ for $1 \leq i \leq n$. This is a *proper prefix* if it does not equal w. An ordered pair of nonempty words (w_1, w_2) in G is a *prefix pair* if w_1 and w_1w_2 are both prefixes of w (note that w_1 will be a proper prefix of w but it may be the case that $w_1w_2 = w$). Recall from Definition 2.1 that given $x \in \mathbb{R}$, we define the *trajectory*

of x under w as the multiset of points:

$$\Omega(w, x) = \{x \cdot \sigma(w_1) \mid w_1 \text{ is a prefix of } w\}.$$

The trajectory of x under w is *stable* if the multiset $\Omega(w, x)$ has no element that occurs more than once, and $x \cdot \sigma(w) = x$.

Let $\{I_n\}_{n\in X}$ be a collection of pairwise disjoint nonempty open intervals in $\mathbb{R}$ for some countable set X which may be finite or infinite. Let $\tau_n : H \to \text{Homeo}^+(I_n), n \in X$ be actions of some group H. We can define a *diagonal-product action*:

$$\tau : H \to \text{Homeo}^+(\mathbb{R}) \qquad \tau := \circledast_{n\in X}\tau_n$$

as the action where each group element fixes each point in $\mathbb{R} \setminus \bigcup_{n\in X} I_n$, and:

$$x \cdot \tau(\alpha) = x \cdot \tau_n(\alpha) \qquad \text{whenever } x \in I_n, \alpha \in H$$

We define the following variation which we also call the diagonal-product, for notational simplicity. Consider an action $\sigma : H \to \text{Homeo}^+(\mathbb{R})$ such that σ pointwise fixes a nonempty open interval I. Consider another action $\tau : H \to \text{Homeo}^+(I)$. Then we define $\sigma \circledast \tau$ also as the diagonal-product action which agrees with σ on $\mathbb{R} \setminus I$ and with τ on I.

Given actions $\sigma : A \to \text{Homeo}^+(\mathbb{R})$ and $\tau : B \to \text{Homeo}^+(\mathbb{R})$, we define the action

$$\eta : A \star B \to \text{Homeo}^+(\mathbb{R}) \qquad \eta := \langle \sigma, \tau \rangle$$

as the action generated by these (using the defining property of free products). We say that η is *generated by the actions* $\sigma$and τ, since it is generated by the images of these actions in $\text{Homeo}^+(\mathbb{R})$. Note that η may not be faithful, even if σ and τ are.

The purpose of this section is to prove Theorem E. If w is a proper power, it is easy to see that a relative stacking cannot exist. So we focus on the hard direction, which is reformulated below.

Theorem 5.1. *Let A, B be countable right-orderable groups and $G = A \star B$. Consider a cyclically reduced word*

$$w = a_1 b_1 \dots a_n b_n \in G, \qquad a_i \in A \setminus \{id\}, b_i \in B \setminus \{id\}.$$

Assume that w is not a proper power. Then there exists an action $\sigma : G \to \text{Homeo}^+(\mathbb{R})$ and some $x \in \mathbb{R}$ such that $\Omega(w, x)$ is stable.

We remark that in Theorem 5.1, we do not require the action σ to be faithful. However, from such a σ, one can easily obtain an action satisfying the same conditions that is faithful. Let $\sigma_1 : G \to \text{Homeo}^+(0, 1)$ be an action obtained by conjugating σ via some homeomorphism $\nu : \mathbb{R} \to (0, 1)$, and let $\sigma_2 : G \to \text{Homeo}^+(1, 2)$ be a faithful action. Note that the latter exists since free products of left orderable groups are left orderable [DNR14]. Then the diagonal-product $\sigma_1 \circledast \sigma_2$ is a faithful action of G that satisfies the requirements of Theorem 5.1, as witnessed by the image $\nu(x)$ of the point x which witnessed this for σ.

The following proposition will be a key step.

Proposition 5.2. *Let A, B be countable right-orderable groups and $G = A \star B$. Consider a cyclically reduced word*

$$w = a_1 b_1 \dots a_n b_n \in A \star B, \qquad a_i \in A \setminus \{id\}, b_i \in B \setminus \{id\}.$$

Assume that w is not a proper power. Let (w_1, w_2) be a prefix pair for w. Then there exists an action of $\sigma : G \to \mathrm{Homeo}^+(\mathbb{R})$ and some $x \in \mathbb{R}$ such that:

(1) $x \cdot \sigma(w) = x$.
(2) $x \cdot \sigma(w_1 w_2) \neq x \cdot \sigma(w_1)$.

First, we will see how we can use Proposition 5.2 to supply the proof of Theorem 5.1. The main ingredient in this will be Lemma 5.3 below, for which we need a few definitions. Let H be a countable group. Given a fixed variable y, we shall consider *systems of equations and inequations over H*, or for brevity simply a *system over H*. Each equation in such a system will be of the form $y \cdot \alpha = y$ and each inequation will be of the form $y \cdot \beta \neq y$ where $\alpha, \beta \in H$ are fixed group elements. A solution to the system will be a pair (τ, x) where $\tau : H \to \mathrm{Homeo}^+(\mathbb{R})$ is an action (not necessarily faithful) and $x \in \mathbb{R}$, so that:

(1) For each equation $y \cdot \alpha = y$ in the system it holds that $x \cdot \tau(\alpha) = x$
(2) For each inequation $y \cdot \beta \neq y$ in the system, it holds that $x \cdot \tau(\beta) \neq x$.

Such a system is called a *finite system* if there are finitely many such equations and inequations. Given a system Λ, we denote by $\Lambda^=$ the set of equations in Λ and by $\Lambda^\neq$ the set of inequations in Λ. Given a finite collection of finite systems $\Lambda_1, \ldots, \Lambda_n$, we define a new system

$$\Xi(\Lambda_1, \ldots, \Lambda_n) := \Big(\bigcup_{1 \leq i \leq n} \Lambda_i^\neq \Big) \cup \Big(\bigcap_{1 \leq i \leq n} \Lambda_i^= \Big).$$

For example, consider the Baumslag–Solitar group $\mathrm{BS}(1,2) = \langle f, g \mid f^{-1} g f = g^2 \rangle$, and consider the system $\Lambda = \{y \cdot f = y, y \cdot g \neq y\}$. Then the action $\tau : \mathrm{BS}(1,2) \to \mathrm{Homeo}^+(\mathbb{R})$ given by $t \cdot \tau(f) = 2t, t \cdot \tau(g) = t + 1$ for $t \in \mathbb{R}$, and the point $0 \in \mathbb{R}$ provides a solution. Indeed, one checks that $0 \cdot \tau(f) = 0$ and $0 \cdot \tau(g) = 1 \neq 0$.

Lemma 5.3. *Given a countable group H, let $\Lambda_1, \ldots, \Lambda_m$ be finite systems over H. If each Λ_i is solvable over H, then $\Xi(\Lambda_1, \ldots, \Lambda_m)$ is solvable over H.*

Proof. We will show this for $m = 2$, and the general case will follow from the same argument using induction, since

$$\Xi(\Lambda_1, \ldots, \Lambda_m) = \Xi(\Xi(\Lambda_1, \ldots, \Lambda_{m-1}), \Lambda_m).$$

Consider actions $\sigma_i : H \to \mathrm{Homeo}^+(\mathbb{R})$ and $x_i \in \mathbb{R}$ that are solutions to the systems Λ_i, for each $1 \leq i \leq 2$. We fix an enumeration of the orbit $O = x_1 \cdot \sigma_1(H)$ as $(p_n)_{n \in \mathbb{N}}$, where $x_1 = p_0$. We perform the "blowup" construction as follows. We replace each point p_n by a closed interval I_n of length $\frac{1}{2^n}$ to obtain a new copy of the real line, upon which the action σ_1 naturally extends to an action $\sigma_1^b : H \to \mathrm{Homeo}^+(\mathbb{R})$ as follows. Whenever $p_i \cdot \sigma_1(\alpha) = p_j$ for some $\alpha \in H$, $\sigma_1^b(\alpha)$ maps the interval I_i to I_j by the unique orientation-preserving affine homeomorphism $\eta_{i,j} : I_i \to I_j$. So the restriction $\sigma_1^b(\alpha)|_{I_i} = \eta_{i,j}$.

The action σ_1^b satisfies the following key property. Recall that we fixed $x_1 = p_0$. For each equation $y \cdot \alpha_1 = y$ and each inequation $y \cdot \alpha_2 \neq y$ in Λ_1, it holds that:

(1) I_0 is fixed pointwise by $\sigma_1^b(\alpha_1)$.
(2) $(I_0 \cdot \sigma_1^b(\alpha_2)) \cap I_0 = \emptyset$.

Since each open interval $int(I_n)$ is homeomorphic to $\mathbb{R}$, we choose homeomorphisms $\nu_n : \mathbb{R} \to int(I_n)$ such that $\nu_i \eta_{i,j} = \nu_j$ for all i, j. Using topological conjugacy given by the maps $\nu_n : \mathbb{R} \to int(I_n)$, for each $n \in \mathbb{N}$ we build an action:

$$\tau_n : H \to \mathrm{Homeo}^+(int(I_n)), \qquad \tau_n = \nu_n^{-1} \sigma_2 \nu_n.$$

We consider the diagonal-product action:

$$\tau : H \to \mathrm{Homeo}^+(\mathbb{R}) \qquad \tau = \circledast_{n \in \mathbb{N}} \tau_n.$$

Note that for each pair $\alpha, \beta \in H$, by construction it holds that

$$\tau(\alpha) \sigma_1^b(\beta) = \sigma_1^b(\beta) \tau(\alpha).$$

This naturally induces an action:

$$\lambda : H \oplus H \to \mathrm{Homeo}^+(\mathbb{R}), \qquad \lambda(\alpha, \beta) = \tau(\alpha) \sigma_1^b(\beta) = \sigma_1^b(\beta) \tau(\alpha).$$

By our hypothesis on σ_2, we have a point $z \in int(I_0)$ such that for each equation $y \cdot \alpha_1 = y$ and inequation $y \cdot \alpha_2 \neq y$ in Λ_2, it holds that

$$z \cdot \tau(\alpha_1) = z, \qquad z \cdot \tau(\alpha_2) \neq z.$$

Let $\rho : H \to H \oplus H$ be the "diagonal embedding" given by $\rho(\alpha) = (\alpha, \alpha)$ for each $\alpha \in H$. Then $\rho\lambda$ is the desired action and z is the required point. □

Proof of Theorem 5.1 assuming Proposition 5.2. We enumerate the set of prefix pairs of w as $\{(w_1^{(i)}, w_2^{(i)}) \mid 1 \leq i \leq k\}$. For each $1 \leq i \leq k$, we define a system Λ_i consisting of one equation and one inequation:

$$y \cdot w = y, \qquad y \cdot w_1^{(i)} w_2^{(i)} (w_1^{(i)})^{-1} \neq y.$$

Applying Proposition 5.2, for each $1 \leq i \leq k$ we obtain a group action $\sigma_i : G \to \mathrm{Homeo}^+(\mathbb{R})$ such that there exists some $x_i \in \mathbb{R}$ for which

$$x_i \cdot \sigma_i(w) = x_i, \qquad x_i \cdot \sigma_i(w_1^{(i)} w_2^{(i)}) \neq x_i \cdot \sigma_i(w_1^{(i)}).$$

Applying Lemma 5.3 to $\Lambda_1, \ldots, \Lambda_k$, we obtain an action $\sigma : G \to \mathrm{Homeo}^+(\mathbb{R})$ which witnesses a solution to the system $\Xi(\Lambda_1, \ldots, \Lambda_k)$ for G as some $x \in \mathbb{R}$. Note that each Λ_i contains the same equation, $y \cdot w = y$, so we have that

$$\Xi(\Lambda_1, \ldots, \Lambda_k) = (\bigcup_{1 \leq i \leq k} \Lambda_i^{\neq}) \cup \{y \cdot w = y\} = \bigcup_{1 \leq i \leq k} \Lambda_i.$$

The point $x \in \mathbb{R}$ that witnesses this for the action σ satisfies that $\Omega(w, x)$ is stable. This proves Theorem 5.1. □

We shall now focus on proving Proposition 5.2. First we simplify it to a proposition that is more convenient to prove. To do so, we make the following basic observation. It generalizes easily to any system over any group, but we only need this for systems consisting of one equation and one inequation. We will also use this in the proof of Proposition 5.5.

Lemma 5.4. *Fix a group H and consider a system Λ with one equation $y \cdot \alpha = y$ and one inequation $y \cdot \beta \neq y$, for some given $\alpha, \beta \in H$. For any $h \in H$, consider the* h-conjugate system Λ' *of Λ with one equation $y \cdot (h^{-1}\alpha h) = y$ and one inequation $y \cdot (h^{-1}\beta h) \neq y$. Then Λ has a solution over H if and only if Λ' does.*

Proof. Suppose Λ has a solution (τ, x), that is, we have $x \cdot \tau(\alpha) = x$ and $x \cdot \tau(\beta) \neq x$. Then $(\tau, x \cdot \tau(h))$ is a solution for Λ'. The other direction works by symmetry. □

Now we show that Proposition 5.2 reduces to the following simpler statement.

Proposition 5.5. *Let A, B be countable right-orderable groups and $G = A \star B$. Consider a cyclically reduced word*

$$w = a_1 b_1 \dots a_n b_n \in A \star B, \qquad a_i \in A \setminus \{id\}, b_i \in B \setminus \{id\},$$

and w_1 a proper prefix of w. Assume that w is not a proper power. Then there exists an action $\tau : A \star B \to \mathrm{Homeo}^+(\mathbb{R})$ and some $x \in \mathbb{R}$ such that $x \cdot \tau(w) = x$ and $x \cdot \tau(w_1) \neq x$.

Proof of Proposition 5.2 assuming Proposition 5.5. The goal is to find a solution (σ, x) for the following system Λ over $G = A \star B$, where we have one equation $y \cdot w = y$ and one inequation $y \cdot (w_1 w_2 w_1^{-1}) \neq y$.

By Lemma 5.4, this is equivalent to solving the w_1-conjugate system Λ' with one equation $y \cdot (w_1^{-1} w w_1) = y$ and one inequation $y \cdot w_2 \neq y$. Note that $w_1^{-1} w w_1$ is a cyclic conjugate of w since w_1 is a prefix of w. And w_2 is a proper prefix of $w_1^{-1} w w_1$, so Λ' has a solution by Proposition 5.5. □

The goal of the rest of the section is to supply the proof of Proposition 5.5. We will make use of the following elementary lemmas.

Lemma 5.6. *Given a nontrivial element f in a countable right-orderable group H, a nonempty open interval I, and $x, y \in I$ with $x < y$, we can find faithful actions $\nu_1, \nu_2 : H \to \mathrm{Homeo}^+(I)$ such that $y < x \cdot \nu_1(f)$ and $y \cdot \nu_2(f) < x$.*

Proof. Since $f \neq id$, for any $s \neq t \in I$, by conjugating a fixed faithful action, there is another faithful action $\nu : H \to \mathrm{Homeo}^+(I)$ such that $s \cdot \nu(f) = t$. The existence of ν_1 and ν_2 easily follows from this by choosing s and t appropriately. □

Lemma 5.7. *Consider a countable right-orderable group H, a nonempty open interval I, and $x, y \in I$ with $x < y$. Let $f, g \in H \setminus \{id\}$ be such that $f \neq g$. Then we can find a faithful action $\sigma : H \to \mathrm{Homeo}^+(I)$ such that $y \cdot \sigma(g) < x \cdot \sigma(f)$.*

Proof. We consider a dynamical realization $\sigma_1 : H \to \mathrm{Homeo}^+(\mathbb{R})$ of H with a free orbit. By conjugating with an orientation-reversing homeomorphism if needed, we can assume that there is a point $z \in \mathbb{R}$ such that $z \cdot \sigma_1(g) < z \cdot \sigma_1(f)$. Using continuity, we find points $x_1 < y_1$ in a small neighborhood of z such that $y_1 \cdot \sigma_1(g) < x_1 \cdot \sigma_1(f)$. Now we consider a homeomorphism $\nu : \mathbb{R} \to I$ such that $\nu(x_1) = x, \nu(y_1) = y$. Upon conjugating σ_1 by ν, we obtain the required σ. □

Before proving Proposition 5.5, first we discuss an elementary case as a warm-up.

Example 5.8. Let A, B be as above and let $w = ab, w_1 = a$ for $a \in A \setminus \{id\}, b \in B \setminus \{id\}$. We would like to build an action $\eta : A \star B \to \mathrm{Homeo}^+(\mathbb{R})$ which witnesses the conclusion of the Proposition 5.5 for these words, that is, $x \cdot \eta(ab) = x$ and $x \cdot \eta(a) \neq x$ for some $x \in \mathbb{R}$.

Let

$$I_1 = [0, 2], J_1 = [1, 3], K = [2, 4], J_1' = [3, 5], I_1' = [4, 6].$$

Using Lemma 5.6, it is straightforward to find actions

$$\sigma_1 : A \to \mathrm{Homeo}^+(I_1), \qquad \tau_1 : B \to \mathrm{Homeo}^+(J_1),$$
$$\sigma_1' : A \to \mathrm{Homeo}^+(I_1'), \qquad \tau_1' : B \to \mathrm{Homeo}^+(J_1'),$$

such that

$$1 \cdot \sigma_1(a) > 1.5, \qquad 1.5 \cdot \tau_1(b) > 2.5,$$
$$5 \cdot \sigma_1'(a) < 4.5, \qquad 4.5 \cdot \tau_1'(b) < 3.5.$$

Now let $\nu : A \to \mathrm{Homeo}^+(K)$ be an action such that $2.5 \cdot \nu(a) > 3.5$. Let

$$\sigma : A \to \mathrm{Homeo}^+[0, 6], \qquad \sigma = \sigma_1 \circledast \nu \circledast \sigma_1',$$

and

$$\tau : B \to \mathrm{Homeo}^+[0, 6], \qquad \tau = \tau_1 \circledast \tau_1'.$$

We define $\eta = \langle \sigma, \tau \rangle$. We call such an action a *dynamical arrangement*, which is formally defined below. For

$$K_1 = [2.5, 3.5], \qquad K_2 = [1, 5],$$

our dynamical arrangement is designed to satisfy that

$$K_1 \cdot \eta(ab) \subset K_2 \cdot \eta(ab) \subset K_1, \qquad (K_1 \cdot \eta(a)) \cap K_1 = \emptyset.$$

In particular, by the intermediate value theorem, ab fixes a point in K_1 and a moves every point in K_1.

Now we introduce some terminology for the general case. For $n \in \mathbb{N}$, a *catenation of intervals* (see Figure 7) is an ordered tuple of the form $(I_1, J_1, \ldots, I_n, J_n)$ or $(I_1, J_1, \ldots, I_n, J_n, I_{n+1})$ of nonempty open intervals in $\mathbb{R}$ such that for all k (in the given range):

(1) Each interval I_k, J_k has length 2, and the endpoints lie in $\mathbb{Z}$;
(2) $\sup(I_k) = \inf(I_{k+1})$ and $\sup(J_k) = \inf(J_{k+1})$; and
(3) $\sup(I_k)$ is the midpoint of J_k and $\inf(J_k)$ is the midpoint of I_k.

Note that a catenation is an ordered tuple of intervals, and in some cases we shall use a different indexing system such as for instance, $(J_m, I_m, \ldots, J_1, I_1)$. However, it will still hold that the intervals appear "from left to right" starting from J_m and onward until I_1.

Given a catenation of the form $(I_1, J_1, \ldots, I_n, J_n)$ or $(I_1, J_1, \ldots, I_n, J_n, I_{n+1})$, a *dynamical arrangement* is an action of $A \star B$ which is generated by the diagonal-product of chosen actions of A on the intervals I_k's and the diagonal-product of chosen actions of B on the intervals J_k's. We shall construct dynamical arrangements that allow us to prove Proposition 5.5.

We record two elementary lemmas concerning actions emerging from catenations. These are essentially natural generalizations of a key idea in the example above.

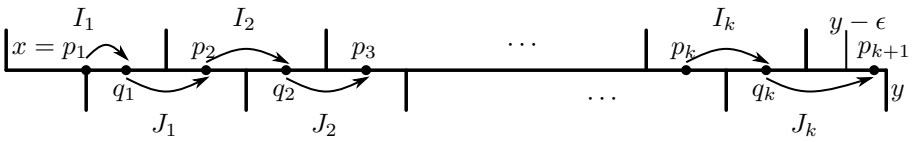

FIGURE 7. The action on a catenation in Lemma 5.9.

Lemma 5.9. *Let A, B be countable right-orderable groups. Consider a reduced word*

$$w = a_1 b_1 \dots a_k b_k \in A \star B, \qquad a_i \in A \setminus \{id\}, b_i \in B \setminus \{id\}.$$

Let $(I_1, J_1, \dots, I_k, J_k)$ be a catenation of intervals. Let $x = \inf(J_1)$, $y = \sup(J_k)$ and let $\epsilon > 0$. We can construct actions

$$\sigma_i : A \to \mathrm{Homeo}^+(I_i), \qquad \tau_i : B \to \mathrm{Homeo}^+(J_i), \qquad 1 \le i \le k,$$

such that the following holds. Let $\nu = \langle \sigma, \tau \rangle$ be the action where

$$\sigma = \circledast_{1 \le i \le k} \sigma_i, \qquad \tau = \circledast_{1 \le i \le k} \tau_i.$$

Then it holds that $x \cdot \nu(a_1 b_1 \dots a_k b_k) \in (y - \epsilon, y)$. Moreover, an analogous statement holds for words of the form $w = a_1 b_1 \dots a_k$.

Proof. Fix $x = p_1$ for notational convenience. First, we construct an action

$$\sigma_1 : A \to \mathrm{Homeo}^+(I_1) \qquad \text{such that } q_1 = p_1 \cdot \sigma_1(a_1) \in int(J_1).$$

Next, we construct an action

$$\tau_1 : B \to \mathrm{Homeo}^+(J_1) \qquad \text{such that } p_2 = q_1 \cdot \tau_1(b_1) \in int(I_2).$$

Proceeding inductively, we construct actions

$$\sigma_i : A \to \mathrm{Homeo}^+(I_i), \qquad \tau_i : B \to \mathrm{Homeo}^+(J_i),$$

such that:

$$q_i = p_i \cdot \sigma_i(a_i) \in int(J_i), \qquad \text{for } 1 \le i \le k.$$

and

$$p_{i+1} = q_i \cdot \tau_i(b_i) \in int(I_{i+1}), \qquad \text{for } 1 \le i < k.$$

Finally, we choose an action

$$\tau_k : B \to \mathrm{Homeo}^+(J_k) \qquad \text{such that } p_{k+1} := q_k \cdot \tau_k(b_k) \in (y - \epsilon, y).$$

See Figure 7. Then the required actions are:

$$\sigma = \circledast_{1 \le i \le k} \sigma_i, \qquad \tau = \circledast_{1 \le i \le k} \tau_i, \qquad \nu = \langle \sigma, \tau \rangle.$$

□

The following lemma is essentially a "mirror image" version of the previous lemma, whose proof is the same via conjugating by an orientation-reversing isometry of $\mathbb{R}$.

Lemma 5.10. *Let A, B be countable right-orderable groups. Consider a reduced word*

$$w = a_1 b_1 \dots a_k b_k \in A \star B, \qquad a_i \in A \setminus \{id\}, b_i \in B \setminus \{id\}.$$

Let $(J_k, I_k, \dots, J_1, I_1)$ be a catenation of intervals. Let $y = \inf(J_k)$, $x = \sup(J_1)$ and let $\epsilon > 0$. We can construct actions

$$\sigma_i : A \to \mathrm{Homeo}^+(I_i), \qquad \tau_i : B \to \mathrm{Homeo}^+(J_i), \qquad 1 \leq i \leq k,$$

such that the following holds. Let $\nu = \langle \sigma, \tau \rangle$ be the action where

$$\sigma = \circledast_{1 \leq i \leq k} \sigma_i, \qquad \tau = \circledast_{1 \leq i \leq k} \tau_i.$$

Then it holds that $x \cdot \nu(a_1 b_1 \dots a_k b_k) \in (y, y + \epsilon)$. Moreover, an analogous statement holds for words of the form $w = a_1 b_1 \dots a_k$. □

We are now ready to provide the final proof of this section.

Proof of Proposition 5.5. The proof consists of two cases.

Case 1: $w_1 = a_1 b_1 \dots a_k$ for some $1 \leq k \leq n$.
Case 2: $w_1 = a_1 b_1 \dots a_k b_k$ for some $1 \leq k < n$.

The proof strategy in both cases is to construct an action $\sigma : G \to \mathrm{Homeo}^+(\mathbb{R})$ for which there is a closed interval I such that

$$I \cdot \sigma(w) \subseteq I, \qquad (I \cdot \sigma(w_1)) \cap I = \emptyset.$$

The former fact implies that w admits a fixed point in I due to the intermediate value theorem, and the latter fact implies that w_1 does not fix any point in I. Upon choosing x to be the w-fixed point in I, this will finish the proof. In the second case, we will have to deal with more technicalities than the first, and that is the only instance where we use the fact that w is not a proper power.

Proof of Case 1: The proof in this case is similar to that of Example 5.8 considered above. Consider a catenation (see Figure 8)

$$(I_1, J_1, \dots, I_n, J_n, L_1, M_1, \dots, M_{k-1}, L_k, J'_n, I'_n, \dots, J'_1, I'_1).$$

We remark that the choice of indices is deliberate. We let:

$$\begin{aligned} x_1 = \inf(J_1), \qquad & x_2 = \sup(J'_1), \\ y_1 = \sup(J_n), \qquad & y_2 = \inf(J'_n), \\ z_1 = y_1 - \frac{1}{4}, \qquad & z_2 = y_2 + \frac{1}{4}. \end{aligned}$$

See Figure 8. Note that $z_1 \in int(J_n \cap L_1)$, $z_2 \in int(J'_n \cap L_k)$.

Applying Lemmas 5.9 and 5.10, we choose faithful group actions

$$\begin{aligned} \nu_i : A \to \mathrm{Homeo}^+(I_i), \qquad \nu'_i : A \to \mathrm{Homeo}^+(I'_i), \qquad 1 \leq i \leq n, \\ \tau_i : B \to \mathrm{Homeo}^+(J_i), \qquad \tau'_i : B \to \mathrm{Homeo}^+(J'_i), \qquad 1 \leq i \leq n, \\ \delta_i : A \to \mathrm{Homeo}^+(L_i), \qquad 1 \leq i \leq k, \\ \pi_i : B \to \mathrm{Homeo}^+(M_i), \qquad 1 \leq i \leq k-1, \end{aligned}$$

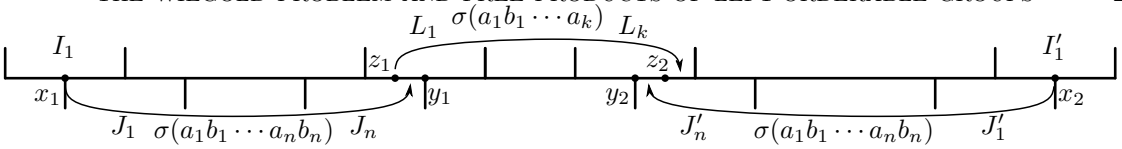

FIGURE 8. The arrangement for Case 1 in the Proof of Proposition 5.5.

and build the action

$$\sigma : G \to \mathrm{Homeo}^+(\mathbb{R}), \qquad \sigma = \langle \eta, \psi \rangle,$$

where

$$\eta = (\circledast_{1\le i\le n}\nu_i) \circledast (\circledast_{1\le i\le k}\delta_i) \circledast (\circledast_{1\le j\le n}\nu_i'),$$
$$\psi = (\circledast_{1\le i\le n}\tau_i) \circledast (\circledast_{1\le i\le k-1}\pi_i) \circledast (\circledast_{1\le j\le n}\tau_i'),$$

and which satisfies (see Figure 8)

(1) $x_1 \cdot \sigma(a_1b_1 \dots a_nb_n) \in (z_1, y_1)$.
(2) $x_2 \cdot \sigma(a_1b_1 \dots a_nb_n) \in (y_2, z_2)$.
(3) $z_1 \cdot \sigma(a_1b_1 \dots a_k) > z_2$.

Let $I = [z_1, z_2]$. We claim that the interval I satisfies the required conditions. From the first two parts and the fact that $x_1 < z_1 < y_1 < y_2 < z_2 < x_2$, it follows that

$$I \cdot \sigma(a_1b_1 \dots a_nb_n) \subseteq I,$$

and hence I contains a fixed point for the element $a_1b_1 \dots a_nb_n$. However, the third part ensures that

$$(I \cdot \sigma(a_1b_1 \dots a_k)) \cap I = \emptyset,$$

and hence no point in I is fixed by $a_1b_1 \dots a_k$. This settles the proof for this case.

Proof of Case 2: First we claim that we can assume without generality that $b_k \neq b_n$. Suppose $b_k = b_n$, then by Lemma 5.4, it suffices to solve the b_k^{-1}-conjugate system, which has one equation $y \cdot (b_n w b_n^{-1}) = y$ and one inequation $y \cdot (b_k w_1 b_k^{-1}) \neq y$. Note that

$$b_n w b_n^{-1} = b_n a_1 b_1 \dots a_n, \quad \text{and} \quad b_k w_1 b_k^{-1} = b_k a_1 b_1 \dots a_k,$$

and the latter is again a proper prefix of the former and of even length. Next we check if $a_n = a_k$ and continue this process. If this never halts, then there is a subword u whose length $|u| = \gcd(|w|, |w_1|)$ such that w and w_1 are both powers of u, contradicting our assumption that w is not a proper power. Thus up to swapping the factors A and B, we may assume without loss of generality that $b_k \neq b_n$.

From now on, assume $b_k \neq b_n$. Consider the catenation

$$(I_1, J_1, \dots, I_n, J_n, L_1, M_1, \dots, M_{k-1}, L_k, J_n', I_n', \dots, J_1', I_1').$$

We let

$$x_1 = \inf(J_1), \qquad x_2 = \sup(J_1'), \qquad y_1 = \sup(J_n) - \frac{1}{2}, \qquad y_2 = \sup(J_n),$$
$$p_1 = \inf(J_n'), \qquad p_2 = \sup(J_n').$$

See Figure 9. Applying Lemmas 5.9 and 5.10, we choose faithful groups actions:

$$\nu_i : A \to \mathrm{Homeo}^+(I_i), \qquad \nu_i' : A \to \mathrm{Homeo}^+(I_i'), \qquad 1 \le i \le n,$$

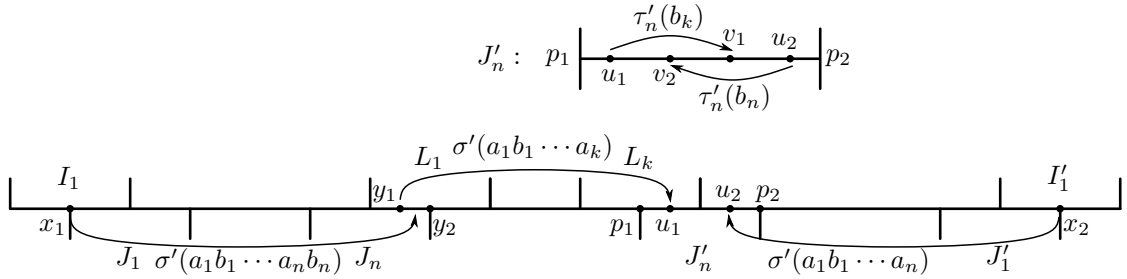

FIGURE 9. The arrangement for Case 2 in the Proof of Proposition 5.5, where the crucial interval J'_n has been magnified. The positions of v_1, v_2 relative to u_1, u_2 are irrelevant, and here we show one possibility.

$$\tau_i : B \to \mathrm{Homeo}^+(J_i), \qquad 1 \le i \le n,$$
$$\tau'_i : B \to \mathrm{Homeo}^+(J'_i), \qquad 1 \le i < n.$$

We emphasize the occurrence of $i < n$ above. In particular, we will choose τ'_n at a later stage. Also choose the following actions using Lemmas 5.9 and 5.10

$$\delta_i : A \to \mathrm{Homeo}^+(L_i), \qquad 1 \le i \le k,$$
$$\pi_i : B \to \mathrm{Homeo}^+(M_i), \qquad 1 \le i \le k-1,$$

and build the action

$$\sigma' : G \to \mathrm{Homeo}^+(\mathbb{R})$$

given by $\sigma' = \langle \eta, \psi \rangle$, where

$$\eta = (\circledast_{1\le i\le n}\nu_i) \circledast (\circledast_{1\le i\le k}\delta_i) \circledast (\circledast_{1\le i\le n}\nu'_i),$$
$$\psi = (\circledast_{1\le i\le n}\tau_i) \circledast (\circledast_{1\le i\le k-1}\pi_i) \circledast (\circledast_{1\le i<n}\tau'_i),$$

and which satisfies (see Figure 9):

$$x_1 \cdot \sigma'(a_1b_1 \dots a_nb_n) \in (y_1, y_2),$$
$$u_1 := y_1 \cdot \sigma'(a_1b_1 \dots a_k) \in (p_1, p_2),$$
$$u_2 := x_2 \cdot \sigma'(a_1b_1 \dots a_n) \in (p_1, p_2).$$

Note that $u_1 < u_2$, since the midpoint of J'_n lies in (u_1, u_2). Using Lemma 5.7, we construct an action $\tau'_n : B \to \mathrm{Homeo}^+(J'_n)$ such that (see the top of Figure 9):

$$v_2 = u_2 \cdot \tau'_n(b_n) < u_1 \cdot \tau'_n(b_k) = v_1$$

Since the action ψ pointwise fixes J'_n by definition, we can define $\psi' = \psi \circledast \tau'_n$. We then define $\sigma = \langle \eta, \psi' \rangle$. We claim that our construction satisfies the requirements for $I = [y_1, v_2]$. To see this, note that

$$x_2 \cdot \sigma(a_1b_1 \dots a_nb_n) = u_2 \cdot \sigma(b_n) = u_2 \cdot \tau'_n(b_n) = v_2,$$

so

$$(x_1, x_2) \cdot \sigma(a_1b_1 \dots a_nb_n) \subset (y_1, v_2).$$

Since $I = [y_1, v_2] \subset (x_1, x_2)$, we deduce that

$$I \cdot \sigma(a_1b_1 \dots a_nb_n) \subseteq I.$$

Moreover,

$$y_1 \cdot \sigma(a_1 b_1 \dots a_k b_k) = u_1 \cdot \sigma(b_k) = u_1 \cdot \tau_n'(b_k) = v_1 > v_2.$$

Therefore, it follows that

$$(I \cdot \sigma(a_1 b_1 \dots a_k b_k)) \cap I = \emptyset.$$

So the interval $I = [y_1, v_2]$ contains a $\sigma(a_1 b_1 \dots a_n b_n)$-fixed point, but no point in I is fixed by $\sigma(a_1 b_1 \dots a_k b_k)$. This settles the proof for this case. □

6. Proof of the main results

Now we have proved the two main ingredients, Theorems D and E. Using them, we first deduce Theorem C.

Proof of Theorem C. Here S has no 2-sphere or disk components, so $-\chi^-(S) = -\chi(S)$.

First consider the case where w is not a proper power. All notions involved are preserved by conjugation, and since w is not conjugate into A or B, we may assume that w is expressed as a cyclically reduced word. Then by Theorem E, there is a relative stacking for w, and hence by Theorem D we have $-\chi^-(S) \geq \deg(S)$. If A or B are uncountable, pass to the countable subgroups A' and B' containing the letters in w, then they are right-orderable since they are subgroups of A and B respectively, so there is a relative stacking for $w \in G' = A' \star B'$, and the result follows using Theorem 4.1 instead of Theorem D.

Now suppose $w = u^k$ is a proper power, where $u \in G$ is not a proper power and $k \geq 2$. Note that u is not conjugate into A or B since w is not, so the conclusion holds for u. Any w-admissible surface S of degree $d = \deg(S)$ is naturally also a u-admissible surface of degree kd, and the notion of boundary-incompressibility is preserved. From what we have shown for u, we obtain $-\chi^-(S) \geq kd = k \deg(S) \geq \deg(S)$. □

Finally, we deduce Theorem B from Theorem C.

Proof of Theorem B. We first show the second assertion, namely, for any $w \in G$ not conjugate into A, the natural map $A \hookrightarrow A \star B$ induces an *injection* $A \hookrightarrow (A \star B)/\langle\langle w \rangle\rangle$. This is obvious if w is conjugate to some $b \in B$ since $(A \star B)/\langle\langle w \rangle\rangle = A \star (B/\langle\langle b \rangle\rangle)$. So we assume w is not conjugate into B either.

Suppose the map is not injective, namely, there is an $a \in A \setminus \{id\}$ such that $a \in \langle\langle w \rangle\rangle$. Then by Example 3.5, there are equations of the form (3.1), and we consider one with minimal length $k \geq 1$. As explained in Example 3.5, this gives rise to a boundary-incompressible w-admissible surface S which is a sphere with $k+1$ boundary components and of degree

$$\deg(S) = \sum_{i=1}^{k} |n_i| \geq k.$$

Note that $-\chi^-(S) = -\chi(S) = (k+1) - 2 = k - 1$. By Theorem C, we get

$$k - 1 = -\chi^-(S) \geq \deg(S) \geq k,$$

which is a contradiction.

This implies that w does not normally generate $A \star B$ when w is not conjugate into A. If w is conjugate to some $a \in A$, then $(A \star B)/\langle\langle w \rangle\rangle = (A/\langle\langle a \rangle\rangle) \star B$, which is nontrivial. Hence $A \star B$ has normal rank greater than 1 for any A and B right-orderable and nontrivial. □

With essentially the same proof, we deduce the following:

Corollary 6.1. *If A and B are right-orderable, and $u \in G = A \star B$ is not conjugate into A or B and is not a proper power, then for any $k \geq 2$, the image of u in $G/\langle\langle u^k \rangle\rangle$ has order k.*

Proof. Denote by $\bar{u}$ the image of u in $G/\langle\langle u^k \rangle\rangle$. Clearly $\bar{u}^k = id$. It remains to show $u^m \notin \langle\langle u^k \rangle\rangle$ whenever m is not divisible by k.

Suppose there is such m with $u^m \in \langle\langle u^k \rangle\rangle$, then we can get an equation

$$u^m = (g_1 u^{kn_1} g_1^{-1})(g_2 u^{kn_2} g_2^{-1}) \cdots (g_\ell u^{kn_\ell} g_\ell^{-1}),$$

where $\ell \in \mathbb{Z}_+$, each $g_i \in G$ and $n_i \neq 0 \in \mathbb{Z}$ for all $1 \leq i \leq \ell$. This yields a surface map $f : S \to X$ for $X = K(G, 1)$ such that S is a sphere with $\ell + 1$ boundary components representing $u^m, u^{kn_1}, \ldots, u^{kn_\ell}$ respectively. This is a u-admissible surface of degree

$$\deg(S) = |m| + k \sum_{i=1}^{\ell} |n_i| \geq k\ell \geq \ell.$$

Among all choices of m not divisible by k and all equations of the form above, choose one with minimal length $\ell \geq 1$. Then the corresponding S is boundary-incompressible by minimality. Thus by Theorem C, we have

$$\ell - 1 = -\chi(S) \geq \deg(S) \geq \ell,$$

which gives a contradiction. □

Under the stronger assumption that A and B are locally indicable, Howie [How82] showed that $A \star B/\langle\langle w \rangle\rangle$ is locally indicable if and only if it is torsion-free and if and only if w is not a proper power. It is natural to ask for an analog, replacing local indicability by right-orderability, and Corollary 6.1 gives a partial implication.

Question 6.2. *If A and B are right-orderable, and $w \in G = A \star B$ is not conjugate into A or B and is not a proper power. Is $G/\langle\langle w \rangle\rangle$ torsion-free? Is it right-orderable?*

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