SPECTRAL GAP OF SCL IN GRAPHS OF GROUPS AND 3-MANIFOLDS

LVZHOU CHEN AND NICOLAUS HEUER

Abstract. Stable commutator length $\mathrm{scl}_G(g)$ of an element g in a group G is an invariant for group elements sensitive to the geometry and dynamics of G.

For any group G acting on a tree, we prove a sharp bound $\mathrm{scl}_G(g) \geq 1/2$ for any g acting without fixed points, provided that the stabilizer of each edge is *relatively torsion-free* in its vertex stabilizers. The sharp gap becomes $1/2 - 1/n$ if the edge stabilizers are *n-relatively torsion-free* in vertex stabilizers. We also compute scl_G for elements acting with a fixed point.

This implies many such groups have a *spectral gap*, that is, there is a constant $C > 0$ such that either $\mathrm{scl}_G(g) \geq C$ or $\mathrm{scl}_G(g) = 0$. New examples include the fundamental group of any 3-manifold using the JSJ decomposition, though the gap must depend on the manifold. We also compute the exact gap of graph products.

We prove these statements by characterizing maps of surfaces to a suitable $K(G, 1)$. In many cases, we also find families of quasimorphisms that realize these gaps. In particular, we construct quasimorphisms realizing the $1/2$-gap for free groups explicitly induced by actions on the circle.

1. Introduction

Let G be a group and let $G' = [G, G]$ be its commutator subgroup. For an element $g \in G'$ we define the *commutator length* ($\mathrm{cl}_G(g)$) of g in G as

$$\mathrm{cl}_G(g) := \min\{n \mid \exists x_1, \dots, x_n, y_1, \dots, y_n \in G : g = [x_1, y_1] \cdots [x_n, y_n]\},$$

and define the *stable commutator length* ($\mathrm{scl}_G(g)$) of g in G as

$$\mathrm{scl}_G(g) := \lim_{n\to\infty} \frac{\mathrm{cl}_G(g^n)}{n}.$$

We extend scl_G to an invariant on the whole group by setting $\mathrm{scl}_G(g) = \mathrm{scl}_G(g^N)/N$ if $g^N \in G'$ for some $N \in \mathbb{Z}_+$ and $\mathrm{scl}_G(g) = \infty$ otherwise. Stable commutator length (scl) arises naturally in geometry, topology and dynamics. See [Cal09b] for an introduction to stable commutator length.

We say that a group G has a *spectral gap in scl* if there is a constant $C > 0$ such that for every $g \in G$ either $\mathrm{scl}_G(g) \geq C$ or $\mathrm{scl}_G(g) = 0$.

The existence of a spectral gap is an obstruction for homomorphisms due to monotonicity of scl. For instance the spectral gap of mapping class groups by [BBF16] implies that any homomorphism from an irreducible lattice of a higher rank semisimple Lie group to a mapping class group has finite image, originally a theorem of Farb–Kaimanovich–Masur [KM96, FM98].

Many classes of groups have spectral gaps, including free groups, word-hyperbolic groups, mapping class groups of closed surfaces, and right-angled Artin groups. See Subsection 2.1 for a list of known results.

In this article we study spectral gaps of groups acting on trees without inversion. By the work of Bass–Serre, such groups may be algebraically decomposed into *graphs of groups* built from their edge and vertex stabilizers; see Subsection 2.3. Basic examples of groups acting on trees are amalgamated free products and HNN extensions. Many classes of groups have a natural graph of groups structure associated to them. Examples include the JSJ decomposition of 3-manifolds

Date: March 4, 2020.

and HNN-hierarchy of one-relator groups, as well as the decomposition of graph products into amalgamated free products.

An element acting on a tree is called *elliptic* if it stabilizes some vertex and *hyperbolic* otherwise. We will discuss the stable commutator length of both types.

We say that a pair of a group G and a subgroup $H \le G$ is *n-relatively torsion-free* (n-RTF) if there is no $1 \le k < n$, $g \in G \setminus H$ and $\{h_i\}_{1\le i\le k} \subset H$ such that

$$gh_1 \cdots gh_k = 1_G,$$

and simply *relatively torsion-free* if we can take $n = \infty$; see Definition 5.4. Similarly, we say that H is *left relatively convex* if there is a G-invariant order on the cosets $G/H = \{gH \mid g \in G\}$ where G acts on the left. Every left relatively convex subgroup is relatively torsion-free; see Lemma 5.15.

Theorem A (Theorems 5.9 and 5.19)**.** *Let G be a group acting on a tree such that the stabilizer of every edge is n-RTF in the stabilizers of the its vertices. If $g \in G$ is hyperbolic, then*

$$\begin{aligned} \mathrm{scl}_G(g) &\ge \frac{1}{2} - \frac{1}{n}, \text{ if } n \in \mathbb{N} \text{ and} \\ \mathrm{scl}_G(g) &\ge \frac{1}{2}, \text{ if } n = \infty. \end{aligned}$$

If the stabilizer of every edge lies left relatively convex in the stabilizers of its vertices, then there is an explicit homogeneous quasimorphism ϕ on G such that $\phi(g) \ge 1$ and $D(\phi) \le 1$.

Our estimates are sharp, strengthening the estimates in [CFL16] and generalizing all other spectral gap results for graph of groups known to the authors [Che18b, DH91, Heu19]. See Subsection 1.1.2 for a stronger version that gives the estimates for individual elements under weaker assumptions.

The stable commutator length generalizes to *chains*, i.e. linear combinations of elements; see Subsection 2.1. We show how to compute scl_G of chains of elliptic elements in terms of the stable commutator length of vertex groups.

Theorem B (Theorem 6.2)**.** *Let G be a group acting on a tree with vertex stabilizers $\{G_v\}$ and let c_v be a chain of elliptic elements in G_v. Then*

$$\mathrm{scl}_G(\sum_v c_v) = \inf \sum_v \mathrm{scl}_{G_v}(c'_v),$$

where each c'_v is a chain of elliptic elements in G_v, and the infimum is taken over all collections $\{c'_v\}$ of chains obtained from adjusting $\{c_v\}$ by chains of elements in edge stabilizers.

See Theorem 6.2 for a precise statement. For example if $G = A \star_{\mathbb{Z}} B$ where $\mathbb{Z}$ is generated by t then we show that $\mathrm{scl}_G(t)$ is the minimum of $\mathrm{scl}_A(t)$ and $\mathrm{scl}_B(t)$; see Theorem 6.8.

We apply our results to obtain spectral gaps of 3-manifold groups using the JSJ decomposition and geometrization theorem.

Theorem C (Theorem 8.11)**.** *For any closed oriented connected 3-manifold M, there is a constant $C(M) > 0$ such that for any $g \in \pi_1(M)$ we have either $\mathrm{scl}_{\pi_1(M)}(g) \ge C(M)$ or $\mathrm{scl}_{\pi_1(M)}(g) = 0$.*

The gap $C(M)$ must depend on M. See Example 6.14. However, we classify elements with $\mathrm{scl}_{\pi_1(M)}(g) = 0$ and describe those with $\mathrm{scl}_{\pi_1(M)}(g) < 1/48$ in Theorem 8.28. See Subsection 1.3 for more details.

1.1. **Method.** Let G be a group and g be an element. There are usually two very different methods of computing $\mathrm{scl}_G(g)$. It is well known that

$$\sup_{\bar{\phi}} \frac{\bar{\phi}(g)}{2D(\bar{\phi})} = \mathrm{scl}_G(g) = \inf_{(f,S)} \frac{-\chi^-(S)}{2n(f)},$$

where $\bar{\phi}$ runs over all *homogeneous quasimorphisms* and where (f, S) runs over all *admissible surfaces*; see Subsection 1.1.1 and 1.1.2. Thus to prove spectral gap results we either need to understand *all* admissible surfaces, or construct *one* good homogeneous quasimorphism for a given element $g \in G$.

1.1.1. *Stable commutator length via quasimorphisms.* A *quasimorphism* on a group G is a map $\phi\colon G \to \mathbb{R}$ such that the *defect* $D(\phi) := \sup_{g,h \in G} |\phi(g) + \phi(h) - \phi(gh)|$ is finite. Two quasimorphisms are *equivalent* if their difference is bounded and a quasimorphism is said to be *homogeneous* if it restricts to a homomorphism on each cyclic subgroup. Bavard's Duality Theorem (Theorem 3.3) asserts that

$$\mathrm{scl}_G(g) = \sup_{\bar{\phi}} \frac{\bar{\phi}(g)}{2D(\bar{\phi})},$$

where the supremum ranges over all homogeneous quasimorphisms. For a given quasimorphism we construct several equivalent quasimorphisms with interesting properties in Section 3. We call a quasimorphism *defect minimizing* if it has minimal defect among all quasimorphisms equivalent to it. We construct special defect minimizing quasimorphisms in the presence of amenability.

Theorem D (Theorem 3.6)**.** *Let $H \leq G$ be an amenable subgroup of a finitely generated group G, suppose that H admits a symmetric Følner sequence and suppose that $\phi\colon G \to \mathbb{R}$ is a quasimorphism. Then there is an equivalent defect minimizing quasimorphism ϕ_H such that for all $g \in G$ and $h \in H$, $\phi_H(gh) = \phi_H(g) + \phi_H(h) = \phi_H(hg)$.*

We will use these maps to compute stable commutator length for edge elements in Section 6.

For the free group on two generators $F(\{\mathtt{a}, \mathtt{b}\})$ we will construct a new family of quasimorphisms which realize the gap of $1/2$, called *circle quasimorphisms.* For what follows, let $\mathcal{A}$ be the set of *alternating words* in the letters $\mathtt{a}$ and $\mathtt{b}$.

Theorem E. *For every element $c \in \mathcal{A}$ there is an explicit quasimorphism*

$$\bar{\mathrm{rot}}_c\colon F(\{\mathtt{a}, \mathtt{b}\}) \to \mathbb{R}$$

constructed using the rotation number induced by an explicit action $\rho_c\colon F(\{\mathtt{a}, \mathtt{b}\}) \to \mathrm{Homeo}^+(S^1)$ on the circle. For every element g in the commutator subgroup of $F(\{\mathtt{a}, \mathtt{b}\})$, there is an element $c \in \mathcal{A}$ such that the quasimorphism $\bar{\mathrm{rot}}_c$ realizes the gap of $1/2$ for g.

This will be described in Section 4. We will use these quasimorphisms to generalize *letter-quasimorphisms* (Definition 4.7) and strengthen the main result of [Heu19], using entirely new methods.

We will construct these letter-quasimorphisms for graph of groups and thus construct new quasimorphisms which realize the gap of $1/2$ for graph of groups provided that the edge groups lie *left relatively convex* in the vertex groups; see Theorem 5.19 in Subsection 5.5.

1.1.2. *Stable commutator length via admissible surfaces.* Let X be a topological space with fundamental group G and let $\gamma\colon S^1 \to X$ be a loop representing $g \in G$. An oriented surface map $f\colon S \to X$ is called *admissible* of degree $n(f) > 0$, if there is a covering map $\partial f : \partial S \to S^1$ of total degree $n(f)$ such that the diagram

$$\begin{tikzcd} \partial S \arrow[r, "\partial f"] \arrow[d, hook] & S^1 \arrow[d, "\gamma"] \\ S \arrow[r, "f"] & X \end{tikzcd}$$

commutes. It is known [Cal09b, Section 2.1] that

$$\mathrm{scl}_G(g) = \inf_{(f,S)} \frac{-\chi^-(S)}{2n(f)},$$

where the infimum ranges over all admissible surfaces and where χ^- is the Euler characteristic ignoring sphere and disk components.

A graph of groups G has a standard realization X that has fundamental group G and contains *vertex and edge spaces* corresponding to the vertex and edge groups. Each hyperbolic element g is represented by some loop γ that cyclically visits finitely many vertex spaces, each time entering the vertex space from one adjacent edge space and exiting from another. A *backtrack* of γ at a vertex space X_v is a time when γ enters and exits X_v from the same edge space X_e. If γ is pulled *tight*, each such a backtrack gives rise to a winding number $g_{e,v} \in G_v \setminus G_e$, where G_v and G_e are the vertex and edge groups corresponding to X_v and X_e. See Subsection 5.1 for more details.

For a subgroup $H \le G$ and $k \ge 2$, an element $g \in G \setminus H$ is *relative k-torsion* if there are $h_1, \dots, h_k \in H$ such that

$$gh_1 \dots gh_k = 1_G.$$

Then H is n-RTF if and only if there is no relative k-torsion for all $k < n$.

In Section 5 we prove the following stronger version of Theorem A.

Theorem A′ (Theorem 5.8)**.** *Let G be a graph of groups. Suppose g is represented by a tight loop γ so that the winding number $g_{e,v}$ associated to any backtrack at a vertex space X_v through an edge space X_e is not relative k-torsion in (G_v, G_e) for any $k < n$. Then*

$$\mathrm{scl}_G(g) \ge \frac{1}{2} - \frac{1}{n}.$$

The proof is based on a *linear programming duality method* that we develop to uniformly estimate the Euler characteristic of all admissible surfaces in X in *normal form.* The normal form is obtained by cutting the surface along edge spaces and simplifying the resulting surfaces, similar to the one in [Che19]. The linear programming duality method is a generalization of the argument for free products in [Che18b].

1.2. **Graphs products.** Let Γ be a simple and not necessarily connected graph with vertex set V and let $\{G_v\}_{v\in V}$ be a collection of groups. The *graph product* G_Γ is the quotient of the free product $\star_{v\in V} G_v$ subject to the relations $[g_u, g_v]$ for any $g_u \in G_u$ and $g_v \in G_v$ such that u, v are adjacent vertices.

Several classes of non-positively curved groups are graph products including right-angled Artin groups; see Example 7.2. We show:

Theorem F (Theorem 7.4)**.** *Let G_Γ be a graph product. Suppose $g = g_1 \cdots g_m \in G_\Gamma$ $(m \ge 1)$ is in cyclically reduced form and there is some $3 \le n \le \infty$ such that $g_i \in G_{v_i}$ has order at least n for all $1 \le i \le k$. Then either*

$$\mathrm{scl}_{G_\Gamma}(g) \ge \frac{1}{2} - \frac{1}{n},$$

or Γ contains a complete subgraph Λ with vertex set $\{v_1, \dots, v_m\}$. In the latter case, we have

$$\mathrm{scl}_{G_\Gamma}(g) = \mathrm{scl}_{G_\Lambda}(g) = \max \mathrm{scl}_{G_i}(g_i).$$

The estimate is sharp: for $g_v \in G_v$ of order $n \geq 2$ and $g_u \in G_u$ of order $m \geq 2$ with u not adjacent to v in Γ, we have $\mathrm{scl}_{G_\Gamma}([g_u, g_v]) = \frac{1}{2} - \frac{1}{\min(m,n)}$; see Remark 7.5. In particular, for a collection of groups with a uniform spectral gap and without 2-torsion, their graph product has a spectral gap.

In the special case of right-angled Artin groups, this provides a new proof of the sharp 1/2 gap [Heu19] that is topological in nature.

1.3. **3-manifold groups.** Let G be the fundamental group of a closed oriented connected 3-manifold M. The prime decomposition of M canonically splits G as a free product, and the JSJ decomposition of 3-manifolds endows each factor group corresponding to a *non-geometric* prime factor the structure of a graph of groups, where each vertex group is the fundamental group of a geometric 3-manifold by the geometrization theorem.

Using this structure, we prove Theorem C for any 3-manifold group in Section 8. This positively answers a question that Genevieve Walsh asked about the existence of spectral gaps of 3-manifolds after a talk by Joel Louwsma.

Although the gap in Theorem C cannot be uniform, its proof implies that elements with scl less than 1/48 must take certain special forms, and it allows us to classify elements with zero scl; see Theorem 8.28. Besides, as is suggested by Michael Hull, prime 3-manifolds only with hyperbolic pieces in the JSJ decomposition have finitely many conjugacy classes with scl strictly less than 1/48 (Corollary 8.29).

For hyperbolic elements in prime factors, we also have a uniform gap 1/48 using the acylindricity of the action and a gap theorem of Clay–Forester–Louwsma [CFL16, Theorem 6.11]. This gap can be improved to 1/6 unless the prime factor contains in its geometric decomposition either the twisted I-bundle over a Klein bottle or a Seifert fibered space over a hyperbolic orbifold that contains cone points of order 2. This is accomplished by using geometry to verify the 3-RTF condition in Theorem A.

The proof of Theorem C relies heavily on estimates of scl in vertex groups (Theorem 8.21). On the one hand this uses a simple estimate in terms of *relative* stable commutator length (Lemma 5.2) together with generalized versions of earlier gap results of hyperbolic groups [Cal08, CF10]. On the other hand, this relies on our characterization of scl in edge groups (Corollary 6.7), where the simple estimate above is useless.

Organization. This article is organized as follows. In Section 2 we recall basic or well known results on stable commutator length and its relative version. In Section 3 we will discuss quasimorphisms and derive nice representatives of extremal quasimorphisms in Subsection 3.2. In Section 4 we describe a new type of extremal quasimorphisms for free groups, which are explicitly induced by an action on a circle. In Section 5 we develop a linear programming duality method to estimate scl of hyperbolic elements in graphs of groups and prove Theorem A. Subsection 5.4 includes a discussion on the crucial n-RTF conditions and key examples. In Section 6 we compute stable commutator length for edge group elements in graphs of groups. We apply our results to obtain spectral gaps in graph products (Section 7) and 3-manifolds (Section 8).

Acknowledgments. We would like to thank Martin Bridson, Danny Calegari and Jason Manning for many helpful suggestions and encouragement. We also thank Lei Chen, Max Forester, Michael Hull, Qixiao Ma, Genevieve Walsh and Yiwen Zhou for useful conversations.

Contents

2. Background

2.1. **Stable commutator length.**

Definition 2.1. Let G be a group and G' its commutator subgroup. Each element $g \in G'$ may be written as a product of commutators $g = [a_1, b_1] \cdots [a_k, b_k]$. The smallest such k is called the *commutator length of* g and denoted by $\mathrm{cl}_G(g)$. The *stable commutator length* is the limit

$$\mathrm{scl}_G(g) := \lim_{n\to\infty} \frac{\mathrm{cl}_G(g^n)}{n}.$$

It is easy to see that cl_G is subadditive and thus that the above limit always exists.

Stable commutator length can be equivalently defined and generalized using *admissible surfaces*.

An integral (rational, or real resp.) *chain* $c = \sum c_i g_i$ is a finite formal sum of group elements $g_1, \ldots, g_m \in G$ with integral (rational, or real resp.) coefficients c_i.

Let X be a space with fundamental group G and let $c = \sum c_i g_i$ be a rational chain. Represent each g_i by a loop $\gamma_i : S^1_i \to X$. An *admissible surface* of degree $n(f) \geq 1$ is a map $f : S \to X$ from a compact oriented surface S with boundary ∂S such that the following diagram commutes and $\partial f[\partial S] = n(f) \sum_i c_i [S^1_i]$,

$$\begin{tikzcd} \partial S \arrow[r, "\partial f"] \arrow[d, hook] & \sqcup S^1_i \arrow[d, "\sqcup \gamma_i"] \\ S \arrow[r, "f"] & X \end{tikzcd}$$

where $\partial S \hookrightarrow S$ is the inclusion map.

Such surfaces exist when the rational chain c is null-homologous, i.e. $[c] = 0 \in H_1(G; \mathbb{Q})$. An admissible surface S is *monotone* if ∂f is a covering map of positive degree on every boundary component of S.

For any *connected* orientable compact surface S, let $\chi^-(S)$ be $\chi(S)$ unless S is a disk or a sphere, in which case we set $\chi^-(S) = 0$. If S is disconnected, we define $\chi^-(S)$ as the sum of $\chi^-(\Sigma)$ over all components Σ of S. Equivalently, $\chi^-(S)$ is the Euler characteristic of S after removing disk and sphere components.

Definition 2.2. For a null-homologous rational chain $c = \sum c_i g_i$ in G as above, its stable commutator length is defined as

$$\mathrm{scl}_G(c) = \inf_S \frac{-\chi^-(S)}{2n(f)},$$

where the infimum is taken over all admissible surfaces S, where $n(f)$ is the corresponding degree. If c is nontrivial in first homology, we make the convention that $\mathrm{scl}_G(c) = +\infty$.

By [Cal09b, Chapter 2], this infimum is the same when considering monotone surfaces. If a chain is a single element $c = g \in G'$, then this agrees with Definition 2.1. For the rest of this paper we will use Definition 2.2 with monotone admissible surfaces.

Let $C_1(G)$ be the space of real chains in G, i.e. the $\mathbb{R}$-vector space with basis G. Let $H(G)$ be the subspace spanned by all elements of the form $g^n - ng$ for $g \in G$ and $n \in \mathbb{Z}$ and $hgh^{-1} - g$ for $g, h \in G$. There is a well defined linear map $h_G : C^H_1(G) \to H_1(G; \mathbb{R})$ sending each chain to its homology class, where $C^H_1(G) := C_1(G)/H(G)$. We denote the kernel by $B^H_1(G)$. Definition 2.2 extends uniquely to null-homologous real chains by continuity. scl vanishes on $H(G)$ and induces a pseudo-norm on $B^H_1(G)$. See [Cal09b, Chapter 2]. Sometimes, scl is a genuine norm, for example if G is word-hyperbolic [CF10].

Here are some basic properties of scl that easily follows from the definitions.

Lemma 2.3.

1. *(Stability)* $\mathrm{scl}_G(g^n) = n \cdot \mathrm{scl}_G(g)$*;*
2. *(Monotonicity) For any homomorphism* $\phi : G \to H$ *and any chain* $c \in C_1(G)$*, we have* $\mathrm{scl}_G(c) \geq \mathrm{scl}_H(\phi(c))$*;*
3. *(Retract) If a subgroup* $H \leq G$ *is a retract, i.e. there is a homomorphism* $r : G \to H$ *with* $r|_H = id$*, then* $\mathrm{scl}_H(c) = \mathrm{scl}_G(c)$ *for all chains* $c \in C_1(H)$*;*
4. *(Direct product) For* $a \in A'$ *and* $b \in B'$ *in the direct product* $G = A \times B$*, we have* $\mathrm{cl}_G(a, b) = \max\{\mathrm{cl}_A(a), \mathrm{cl}_B(b)\}$ *and* $\mathrm{scl}_G(a, b) = \max\{\mathrm{scl}_A(a), \mathrm{scl}_B(b)\}$*.*

Note that finite-order elements can be removed from a chain without changing scl. Thus we will often assume elements in chains to have infinite order.

Definition 2.4. A group G has a *spectral gap* $C>0$ if for any $g\in G$ either $\mathrm{scl}_G(g)\ge C$ or $\mathrm{scl}_G(g)=0$. If in addition, the case $\mathrm{scl}_G(g)=0$ only occurs when g is torsion, we say G has a *strong spectral gap* C.

Many classes of groups are known to have a spectral gap.

(1) G trivially has a spectral gap C for any $C>0$ if scl_G vanishes on $B_1^H(G)$, which is the case if G is amenable [Cal09b, Theorem 2.47] or an irreducible lattice in a semisimple Lie group of higher rank [BM99, BM02]; the gap is strong if G is abelian;
(2) G has a strong spectral gap $1/2$ if G is residually free [DH91];
(3) δ-hyperbolic groups have gaps depending on the number of generators and δ [CF10]; the gap is strong if the group is also torsion-free;
(4) Any finite index subgroup of the mapping class group of a (possibly punctured) closed surface has a spectral gap [BBF16];
(5) All right-angled Artin groups have a strong gap $1/2$ [Heu19] (see [FFT19, FSTar] for earlier weaker estimates);
(6) All Baumslag–Solitar groups have a gap $1/12$ [CFL16].

Our Theorem C adds all 3-manifold groups to the list.

The gap property is essentially preserved under taking free products.

Lemma 2.5 (Clay–Forester–Louwsma)**.** *Let $G=\star_\lambda G_\lambda$ be a free product. Then for any $g\in G$ not conjugate into any factor, we have either $\mathrm{scl}_G(g)=0$ or $\mathrm{scl}_G(g)\ge 1/12$. Moreover, $\mathrm{scl}_G(g)=0$ if and only if g is conjugate to g^{-1}. Thus if the groups G_λ have a uniform spectral gap $C>0$, then G has a gap $\min\{C,1/12\}$.*

Proof. If g does not conjugate into any factor, then it either satisfies the so-called well-aligned condition in [CFL16] or is conjugate to its inverse. In this case, it follows from [CFL16, Theorem 6.9] that either $\mathrm{scl}_G(g)\ge 1/12$ or $\mathrm{scl}_G(g)=0$, corresponding to the two situations. Assuming the factors have a uniform gap C, if an element $g\in G$ conjugates into some factor G_λ, then $\mathrm{scl}_G(g)=\mathrm{scl}_{G_\lambda}(g)\ge C$ since G_λ is a retract of G. □

The constant $1/12$ is optimal in general, but can be improved if there is no torsion of small order. See [Che18b] or [IK18].

Many other groups have a uniform positive lower bound on *most* elements. They often satisfy a spectral gap in a relative sense, which we will introduce in the next subsection.

2.2. **Relative stable commutator length.** We will use relative stable commutator length to state our results in the most natural and the strongest form. It was informally mentioned or implicitly used in [CF10, CFL16, IK18], and formally formulated and shown to be useful in scl computations in [Che19].

Definition 2.6. Let $\{G_\lambda\}_{\lambda\in\Lambda}$ be a collection of subgroups of G. Let $C_1(\{G_\lambda\})$ be the subspace of $C_1(G)$ consisting of chains of the form $\sum_\lambda c_\lambda$ with $c_\lambda\in C_1(G_\lambda)$, where all but finitely many c_λ vanish in each summation.

For any chain $c\in C_1(G)$, define its *relative stable commutator length* to be

$$\mathrm{scl}_{(G,\{G_\lambda\})}(c):=\inf\{\mathrm{scl}_G(c+c'):c'\in C_1(\{G_\lambda\})\}.$$

Let $H_1(\{G_\lambda\})\le H_1(G;\mathbb{R})$ be the subspace of homology classes represented by chains in $C_1(\{G_\lambda\})$. Recall that we have a linear map $h_G: C_1^H(G)\to H_1(G)$ taking chains to their homology classes. Denote $B_1^H(G,\{G_\lambda\}):=h_G^{-1}H_1(\{G_\lambda\})$, which contains $B_1^H(G)$ as a subspace. Then $\mathrm{scl}_{(G,\{G_\lambda\})}$ is finite on $B_1^H(G,\{G_\lambda\})$ and is a pseudo-norm.

The following basic properties of relative scl will be used later.

Lemma 2.7. *Let G be a group and $\{G_\lambda\}$ be a collection of subgroups.*

(1) $\mathrm{scl}_G(c) \ge \mathrm{scl}_{(G,\{G_\lambda\})}(c)$ *for any* $c \in C_1^H(G)$.
(2) If g^n *conjugates into some* G_λ *for some integer* $n \neq 0$, *then* $\mathrm{scl}_{(G,\{G_\lambda\})}(g) = 0$.
(3) (Stability) For any $g \in G$, *we have* $\mathrm{scl}_{(G,\{G_\lambda\})}(g^n) = n \cdot \mathrm{scl}_{(G,\{G_\lambda\})}(g)$.
(4) (Monotonicity) Let $\phi : G \to H$ *be a homomorphism such that* $\phi(G_\lambda) \subset H_\lambda$ *for a collection of subgroups* H_λ *of* H, *then for any* $c \in C_1(G)$ *we have*
$$\mathrm{scl}_{(G,\{G_\lambda\})}(c) \ge \mathrm{scl}_{(H,\{H_\lambda\})}(\phi(c)).$$

For rational chains, relative scl can be computed using *relative admissible surfaces*, which are admissible surfaces possibly with extra boundary components in $\{G_\lambda\}$. This is [Che19, Proposition 2.9] stated as in Lemma 2.9 below.

Definition 2.8. Let $c \in B_1^H(G, \{G_\lambda\})$ be a rational chain. A surface S together with a specified collection of boundary components $\partial_0 \subset \partial S$ is called *relative admissible* for c of degree $n > 0$ if ∂_0 represents $[nc] \in C_1^H(G)$ and every other boundary component of S represents an element conjugate into G_λ.

Lemma 2.9 ([Che19])**.** *For any rational chain* $c \in B_1^H(G, \{G_\lambda\})$, *we have*
$$\mathrm{scl}_{(G,\{G_\lambda\})}(c) = \inf \frac{-\chi^-(S)}{2n},$$
where the infimum is taken over all relative admissible surfaces for c.

The Bavard duality is a dual description of scl in terms of *quasimorphisms*; see Subsection 3.1 and Theorem 3.3. It naturally generalizes to relative stable commutator length.

Lemma 2.10 (Relative Bavard's Duality)**.** *For any chain* $c \in B_1^H(G, \{G_\lambda\})$, *we have*
$$\mathrm{scl}_{(G,\{G_\lambda\})}(c) = \sup \frac{f(c)}{2D(f)},$$
where the supremum is taken over all homogeneous quasimorphisms f *on* G *that vanish on* $C_1(\{G_\lambda\})$.

Proof. Denote the space of homogeneous quasimorphisms on G by $\mathcal{Q}(G)$. Let $N(G)$ be the subspace of $B_1(G)$ where scl vanishes. Then the quotient $B_1(G)/N(G)$ with induced scl becomes a normed vector space. Denote the quotient map by $\pi : B_1(G) \to B_1(G)/N(G)$. A more precise statement of Bavard's duality Theorem 3.3 shows that the dual space of $B_1(G)/N(G)$ is exactly $\mathcal{Q}(G)/H^1(G)$ equipped with the norm $2D(\cdot)$; see [Cal09b, Sections 2.4 and 2.5]. Then scl further induces a norm $\|\cdot\|$ on the quotient space V of $B_1(G)/N(G)$ by the closure of $\pi(C_1(\{G_\lambda\}) \cap B_1(G))$. By definition we have $\|\bar{c}\| = \mathrm{scl}_{(G,\{G_\lambda\})}(c)$ for any $c \in B_1(G)$, where $\bar{c}$ is the image in V. It is well known that the dual space of V is naturally isomorphic to the subspace of $\mathcal{Q}(G)/H^1(G)$ consisting of linear functionals that vanish on $C_1(\{G_\lambda\}) \cap B_1(G)$. Any $\bar{f} \in \mathcal{Q}(G)/H^1(G)$ with this vanishing property can be represented by some $f \in \mathcal{Q}(G)$ that vanishes on $C_1(\{G_\lambda\})$. This proves the assertion assuming $c \in B_1^H(G)$. The general case easily follows since any $c \in B_1^H(G, \{G_\lambda\})$ can be replaced by $c + c' \in B_1^H(G)$ for some $c' \in C_1(\{G_\lambda\})$ without changing both sides of the equation. □

Definition 2.11. For a collection of subgroups $\{G_\lambda\}$ of G and a positive number C, we say $(G, \{G_\lambda\})$ has a *strong relative spectral gap* C if either $\mathrm{scl}_{(G,\{G_\lambda\})}(g) \ge C$ or $\mathrm{scl}_{(G,\{G_\lambda\})}(g) = 0$ for all $g \in G$, where the latter case occurs if and only if g^n conjugates into some G_λ for some $n \neq 0$.

Some previous work on spectral gap properties of scl can be stated in terms of or strengthened to strong relative spectral gap.

Theorem 2.12.

(1) [Che18b, Theorem 3.1] *Let $n \geq 3$ and let $G = \star_\lambda G_\lambda$ be a free product where G_λ has no k-torsion for all $k < n$. Then $(G, \{G_\lambda\})$ has a strong relative spectral gap $\frac{1}{2} - \frac{1}{n}$.*

(2) [Heu19, Theorem 6.3] *Suppose we have inclusions of groups $C \hookrightarrow A$ and $C \hookrightarrow B$ such that both images are left relatively convex subgroups (see Definition 5.13). Let $G = A \star_C B$ be the associated amalgam. Then $(G, \{A, B\})$ has a strong relative spectral gap $\frac{1}{2}$.*

(3) [CF10, Theorem A′] *Let G be δ-hyperbolic with symmetric generating set S. Let a be an element with $a^n \neq ba^{-n}b^{-1}$ for all $n \neq 0$ and all $b \in G$. Let $\{a_i\}$ be a collection of elements with translation lengths bounded by T. Suppose a^n does not conjugate into any $G_i := \langle a_i \rangle$ for any $n \neq 0$, then there is $C = C(\delta, |S|, T) > 0$ such that $\mathrm{scl}_{(G,\{G_i\})}(a) \geq C$.*

(4) [Cal08, Theorem C] *Let M be a compact 3-manifold with tori boundary (possibly empty). Suppose the interior of M is hyperbolic with finite volume. Then $(\pi_1 M, \pi_1 \partial M)$ has a strong relative spectral gap $C(M) > 0$, where $\pi_1 \partial M$ is the collection of peripheral subgroups.*

Proof. Our Theorem 5.9 immediately implies (1) and (2). Part (3) is an equivalent statement of the original theorem [CF10, Theorem A′] in view of Lemma 2.10.

Part (4) is stated stronger than the original form [Cal08, Theorem C] but can be proved in the same way. See Theorem 8.10. □

2.3. **Graphs of groups.** Let Γ be a connected graph with vertex set V and edge set E. Each edge $e \in E$ is oriented with origin $o(e)$ and terminus $t(e)$. Denote the same edge with opposite orientation by $\bar{e}$, which provides an involution on E satisfying $t(\bar{e}) = o(e)$ and $o(\bar{e}) = t(e)$.

A *graph of groups* with underlying graph Γ is a collection of *vertex groups* $\{G_v\}_{v \in V}$ and *edge groups* $\{G_e\}_{e \in E}$ with $G_e = G_{\bar{e}}$, as well as injections $t_e : G_e \hookrightarrow G_{t(e)}$ and $o_e : G_e \hookrightarrow G_{o(e)}$ satisfying $t_{\bar{e}} = o_e$. Let (X_v, b_v) and (X_e, b_e) be pointed $K(G_v, 1)$ and $K(G_e, 1)$ spaces respectively, and denote again by t_e, o_e the maps between spaces inducing the given homomorphisms t_e, o_e on fundamental groups. Let X be the space obtained from the disjoint union of $\sqcup_{e \in E} X_e \times [-1, 1]$ and $\sqcup_{v \in V} X_v$ by gluing $X_e \times \{1\}$ to $X_{t(e)}$ via t_e and identifying $X_e \times \{s\}$ with $X_{\bar{e}} \times \{-s\}$ for all $s \in [-1, 1]$ and $e \in E$. We refer to X as the *standard realization* of the graphs of groups and denote its fundamental group by $G = \mathcal{G}(\Gamma, \{G_v\}, \{G_e\})$. This is called the *fundamental group of the graph of groups*. When there is no danger of ambiguity, we will simply refer to G as the graph of groups.

In practice, we will choose a preferred orientation for each unoriented edge $\{e, \bar{e}\}$ by working with e and ignoring $\bar{e}$.

Example 2.13.

(1) Let Γ be the graph with a single vertex v and an edge $\{e, \bar{e}\}$ connecting v to itself. Let $G_e \cong G_v \cong \mathbb{Z}$. Fix nonzero integers m, ℓ, and let the edge inclusions $o_e, t_e : G_e \hookrightarrow G_v$ be given by $o_e(1) = m$ and $t_e(1) = \ell$. Let the edge space X_e and vertex space X_v be circles S^1_e and S^1_v respectively. Then the standard realization X is obtained by gluing the two boundary components of a cylinder $S^1_e \times [-1, 1]$ to the circle S^1_v wrapping around m and ℓ times respectively. See the left of Figure 1. The fundamental group is the *Baumslag–Solitar group* $\mathrm{BS}(m, \ell)$, which has presentation

$$\mathrm{BS}(m, \ell) = \langle a, t \mid a^m = t a^\ell t^{-1} \rangle.$$

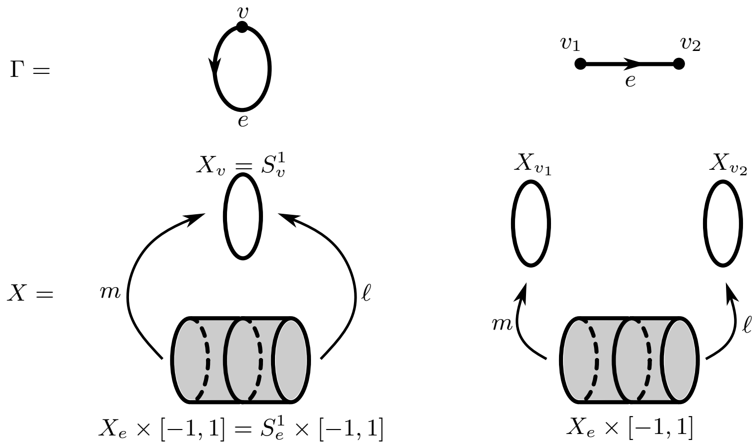

FIGURE 1. On the left we have the underlying graph Γ and the standard realization X of $\mathrm{BS}(m,\ell)$; on the right we have the graph Γ and the realization X for the amalgam $\mathbb{Z}\star_{\mathbb{Z}}\mathbb{Z}$ associated to $\mathbb{Z}\xrightarrow{\times m}\mathbb{Z}$ and $\mathbb{Z}\xrightarrow{\times \ell}\mathbb{Z}$.

In general, with the same graph Γ, for any groups $G_e = C$ and $G_v = A$ together with two inclusions $t_e, o_e : C \hookrightarrow A$, the corresponding graph of groups is the *HNN extension* $G = A_{\star C}$.

(2) Similarly, if we let Γ be the graph with a single edge $\{e, \bar{e}\}$ connecting two vertices $v_1 = o(e)$ and $v_2 = t(e)$, the graph of groups associated to two inclusions $t_e : G_e \to G_{v_1}$ and $o_e : G_e \to G_{v_2}$ is the *amalgam* $G_{v_1} \star_{G_e} G_{v_2}$. See the right of Figure 1 for an example where all edge and vertex groups are $\mathbb{Z}$.

In general, each connected subgraph of Γ gives a graph of groups, whose fundamental group *injects* into G, from which we see that each separating edge of Γ splits G as an amalgam and each non-separating edge splits G as an HNN extension. Hence G arises as a sequence of amalgamations and HNN extensions.

It is a fundamental result of the Bass–Serre theory that there is a correspondence between groups acting on trees (without inversions) and graphs of groups, where vertex and edge stabilizers correspond to vertex and edge groups respectively. See [Ser80] for more details about graphs of groups and their relation to groups acting on trees.

In the standard realization X, the homeomorphic images of $X_e \times \{0\} \cong X_e$ and X_v are called an *edge space* and a *vertex space* respectively. The image of $X_v \sqcup (\sqcup_{e:t(e)=v} X_e \times [0,1))$ deformation retracts to X_v. We refer to its completion $N(X_v)$ as the *thickened vertex space*; see Figure 2.

Free homotopy classes of loops in X fall into two types. *Elliptic* loops are those admitting a representative supported in some vertex space, and such a representative is called a *tight* elliptic loop. Loops of the other type are called *hyperbolic*. We can deform any hyperbolic loop γ so that, for each $s \in (-1,1)$ and $e \in X_e$, γ is either disjoint from $X_e \times \{s\}$ or intersects it only at $\{b_e\} \times \{s\}$ transversely. For such a representative, the edge spaces cut γ into finitely many arcs, each supported in some thickened vertex spaces $N(X_v)$. The image of each arc α in $N(X_v)$ under the deformation retraction $N(X_v) \to X_v$ becomes a based loop in X_v and thus represents an element $w(\alpha) \in G_v$. If some arc α enters and leaves X_v via the same end of an edge e with $t(e) = v$ and $w(\alpha) \in t_e(G_e)$, then we say γ trivially backtracks at α and can pushed α off X_v to further simplify γ; see Figure 3. After finitely many such simplifications, we may assume γ does not trivially backtrack. Refer to such a representative as a *tight* hyperbolic loop, which exists in each hyperbolic homotopy class by the procedure above.

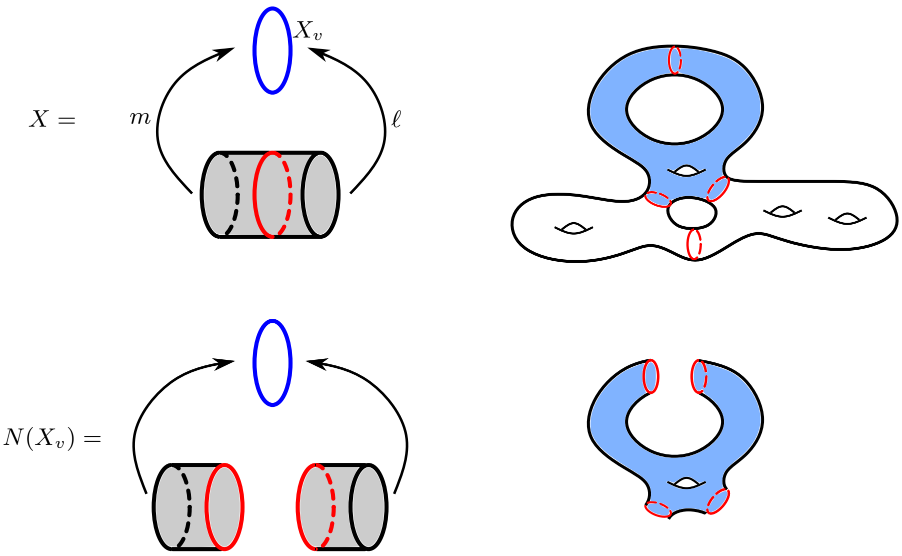

FIGURE 2. On the bottom left we depict the thickened vertex space $N(X_v)$ for the unique vertex space X_v in the realization X (upper left) of $\mathrm{BS}(m, \ell)$; on the right depict a thickened vertex space in a more general situation, where the red circles are the edge spaces that we cut along.

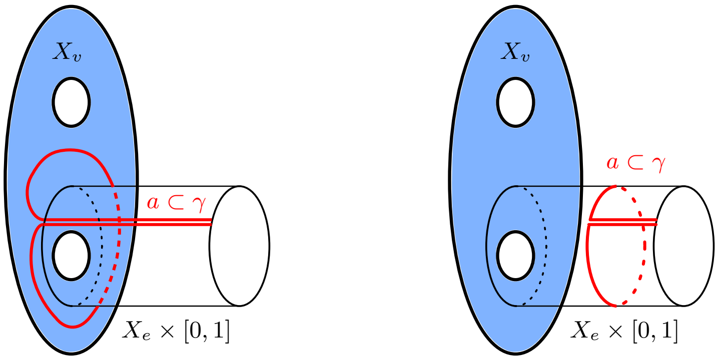

FIGURE 3. A loop γ trivially backtracks at an arc a supported in the thickened vertex space $N(X_v)$ as shown on the left. It can be pushed off the vertex space X_v by a homotopy as shown on the right.

On the group theoretic side, an element $g \in G$ is elliptic (resp. hyperbolic) if it is represented by an elliptic (resp. hyperbolic) loop in X, which we usually choose to be tight. Then an element is elliptic if and only if it conjugates into some vertex group.

2.4. **Circle actions and Euler class.** We recall the classical connection between circle actions and bounded cohomology. See [Ghy87] and also see [BFH16] for a thorough treatment of this topic.

Let G be a group and let V be $\mathbb{Z}$ or $\mathbb{R}$ the *bounded cohomology with coefficients in* V is the homology of the resolution $(C_b^n(G, V), \delta^n)$ of bounded functions $C_b^n(G, V)$. It may be computed both via the homogeneous or inhomogeneous resolution; see [Fri17] for details. In this subsection we will use both types of resolution and always indicate which type we are using.

For three points $x_1, x_2, x_3 \in S^1$ on the circle let $\mathrm{Or}(x_1, x_2, x_3) \in \{-1, 0, 1\}$ be the respective orientation of those three points. Now fix a point $\xi \in S^1$ and define the map $\mathrm{eu} \in C^2_b(\mathrm{Homeo}^+(S^1), \mathbb{Z})$ (in homogeneous resolution) by setting

$$\mathrm{eu}(g_1, g_2, g_3) = \mathrm{Or}(g_1.\xi, g_2.\xi, g_3.\xi).$$

It is a well known fact that eu is a (bounded) cocycle and that the corresponding class [eu] in $H^2_b(\mathrm{Homeo}^+(S^1), \mathbb{Z})$ and $H^2(\mathrm{Homeo}^+(S^1), \mathbb{Z})$ is independent of the basepoint ξ. We call eu the *Euler cocycle.* Here, $H^*_b(\cdot, \cdot)$ denotes the *bounded cohomology.* See [Fri17] for an introduction to bounded cohomology of free groups.

Let $\mathrm{Homeo}^+(\mathbb{R})$ be the group defined via

$$\mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}) = \{\phi \colon \mathbb{R} \to \mathbb{R} \mid \phi \in \mathrm{Homeo}^+(\mathbb{R}) \text{ and commutes with } \tau\},$$

where $\tau(n) \colon \mathbb{R} \to \mathbb{R}$ is the shift map $\tau \colon x \mapsto x + 1$. It can be seen that the class $[\mathrm{eu}] \in H^2(\mathrm{Homeo}^+(S^1); \mathbb{Z})$ corresponds to the central extension

$$0 \to \mathbb{Z} \to \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}) \to \mathrm{Homeo}^+(S^1) \to 1$$

where $\mathbb{Z} \to \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$ is defined via $n \mapsto \tau^n$.

On $\mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$ we may define the *rotation number* $\mathrm{rot} \colon \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}) \to \mathbb{R}$ via

$$\mathrm{rot} \colon g \to \lim_{n \to \infty} \frac{g^n.\xi}{n}.$$

It is well known that the limit exists and that it is independent of the point $\xi \in \mathbb{R}$.

Theorem 2.14 ([Cal09b, Theorem 2.43])**.** *The map* rot *is a homogeneous quasimorphism of defect* 1*. Moreover we have that*

$$\mathrm{scl}(g) = \frac{\mathrm{rot}(g)}{2}$$

for every $g \in \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$.

Now let G be a group acting on the circle via $\rho \colon G \to \mathrm{Homeo}^+(S^1)$. The pullback $\rho^*\mathrm{eu}$ of the Euler cocycle via ρ defines a class $[\rho^*\mathrm{eu}] \in H^2_b(G; \mathbb{Z})$. If this class vanishes under the comparison map $c^2 \colon H^2_b(G; \mathbb{Z}) \to H^2(G; \mathbb{Z})$ then the action $\rho \colon G \to \mathrm{Homeo}^+(S^1)$ can be lifted to an action $\tilde{\rho} \colon G \to \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$. Then define the quasimorphism ϕ via

$$\phi = \tilde{\rho}^*\mathrm{rot}.$$

It can be seen that $\delta\phi = [\rho^*\mathrm{eu}] \in H^2_b(G; \mathbb{Z})$, where $\delta\phi$ denotes the coboundary of the function map ϕ in inhomogeneous resolution.

3. Quasimorphisms

Let G be a group. A *quasimorphism* is a map $\phi \colon G \to \mathbb{R}$ for which there is a constant D, such that

$$|\phi(g) + \phi(h) - \phi(gh)| \leq D$$

for all $g, h \in G$. The infimum of all such D is called the *defect* of ϕ and denoted by $D(\phi)$. Quasimorphisms form a vector space under pointwise addition and scalar multiplication. Trivial examples of quasimorphisms include homomorphisms to $\mathbb{R}$ and bounded functions.

For a function $\beta \colon G \to \mathbb{R}$ we set $\|\beta\| \in \mathbb{R} \cup \{\infty\}$ via $\|\beta\| = \sup_{g \in G} |\beta(g)|$. We say that two quasimorphisms ϕ and ψ are *equivalent* if $\|\phi - \psi\| < \infty$. We say that a quasimorphism is *defect minimizing* if the defect $D(\phi)$ is minimal among all quasimorphisms equivalent to it.

For a fixed quasimorphism ϕ there are several choices of equivalent quasimorphisms with interesting properties, which we will explore in this section:

(1) The homogenization $\bar{\phi}$ is the unique equivalent quasimorphism which restricts to a homomorphism on cyclic subgroups. These maps may be used to compute stable commutator length (Theorem 3.3) and restrict to homomorphisms on amenable subgroups. Compare also Lemma 2.10. Such representatives are *not* defect minimizing; see Subsection 3.1.
(2) The harmonification $\tilde{\phi}$ is the unique equivalent quasimorphism which is bi-harmonic with respect to a certain fixed measure. These representatives are defect minimizing; see Subsection 3.2.
(3) For a certain fixed amenable subgroup $H \leq G$ we will construct an equivalent quasimorphism $\tilde{\phi}_H$ which *both* restricts to a homomorphism on H and is defect minimizing; see Subsection 3.3. These maps are not necessarily unique.

3.1. **Homogeneous quasimorphisms and Bavard's duality theorem.** A quasimorphism $\phi\colon G \to \mathbb{R}$ is said to be *homogeneous* if for all $g \in G$ and $n \in \mathbb{Z}$ we have $\phi(g^n) = n \cdot \phi(g)$.

Let $\phi\colon G \to \mathbb{R}$ be a quasimorphism. Then there is an associated equivalent homogeneous quasimorphism $\bar{\phi}\colon G \to \mathbb{R}$ defined by setting

$$\bar{\phi}(g) := \lim_{n\to\infty} \frac{\phi(g^n)}{n}.$$

We call $\bar{\phi}$ the *homogenization* of ϕ. Homogeneous quasimorphisms on G form a vector space, denoted by $\mathcal{Q}(G)$.

Proposition 3.1 (Homogeneous Representative, [Cal09b, Lemma 2.58])**.** *Let $\phi\colon G \to \mathbb{R}$ be a quasimorphism and let $\bar{\phi}\colon G \to \mathbb{R}$ be its homogenization. Then $\bar{\phi}$ is the unique homogeneous quasimorphism equivalent to ϕ with $\|\phi - \bar{\phi}\| \leq D(\phi)$. The defect satisfies $D(\bar{\phi}) \leq 2D(\phi)$.*

In this section, we will decorate quasimorphisms with a bar-symbol to indicate that they are homogeneous, even if they are not explicitly induced by a non-homogeneous quasimorphism. We collect some well known properties for homogeneous quasimorphisms:

Proposition 3.2 (Homogeneous Quasimorphisms Properties, [Cal09b])**.** *Let $\bar{\phi}\colon G \to \mathbb{R}$ be a homogeneous quasimorphism. Then $\bar{\phi}$ is invariant under conjugation and restricts to a homomorphism on each amenable subgroup $H \leq G$.*

Bavard's Duality Theorem provides the connection to stable commutator length.

Theorem 3.3 (Bavard's Duality Theorem [Bav91])**.** *Let G be a group and let $g \in G$. Then*

$$\mathrm{scl}(g) = \sup_{\bar{\phi}} \frac{|\bar{\phi}(g)|}{2D(\bar{\phi})},$$

where the supremum is taken over all homogeneous quasimorphisms $\bar{\phi}$ on G.

For a fixed $g \in G$ the supremum is achieved by an extremal *quasimorphism $\bar{\phi}$ which is equivalent to a defect minimizing quasimorphism ϕ which satisfies $2D(\phi) = D(\bar{\phi})$.*

Extremal quasimorphisms are notoriously hard to construct. For free groups, explicit constructions of extremal quasimorphisms are known for words with scl value $1/2$ but not in general; see [Heu19] and also [Cal09a, CFL16].

3.2. **Harmonic quasimorphisms.** In this subsection we restrict our attention to finitely generated groups although it is possible to define bi-harmonic quasimorphisms in greater generality.

Let G be a finitely generated group with discrete topology and let μ be a probability measure on G. We say that μ is *symmetric* if $\iota^*\mu = \mu$ for $\iota\colon G \to G$ the involution $\iota\colon g \mapsto g^{-1}$. The *support* of a measure μ is the set

$$\operatorname{supp}(\mu) := \{g \in G \mid \mu(\{g\}) \neq 0\}$$

and we say that a measure is *non-degenerate* if $\operatorname{supp}(\mu)$ generates G. Let d_S be the distance in the Cayley graph associated to any finite generating set S of G. A measure is said to have *finite first moment* if the sum

$$\sum_{g \in G} \mathrm{d}_S(1, g)\mu(\{g\})$$

converges. Note that this definition is independent of the choice of finite generating set S.

Following [BH11] we call a quasimorphism $\phi\colon G \to \mathbb{R}$ *right-harmonic* if $\phi(g) = \sum_{h \in G} \phi(gh)\mu(h)$ for every $g \in G$ and left-harmonic if $\phi(g) = \sum_{h \in G} \phi(hg)\mu(h)$ for every $g \in G$.

Theorem 3.4 (Bi-Harmonic Representatives, [Hub13, Theorem 2.16] [BH11, Proposition 2.2])**.** *Let G be a finitely generated group and let μ be a symmetric, non-degenerate probability measure on G with finite first moment. For every quasimorphism ϕ there is a unique equivalent bi-harmonic quasimorphism $\tilde{\phi}$, which satisfies*

$$\|\tilde{\phi} - \phi\| \leq 3D(\phi).$$

Moreover, $\tilde{\phi}$ is defect minimizing, and in particular we have

$$D(\tilde{\phi}) \leq D(\phi).$$

Proof. The existence and uniqueness of $\tilde{\phi}$ was proven first by Burger and Monod in [BM99, Corollary 3.14]. Huber showed in [Hub13, Theorem 2.16] that $\tilde{\phi}$ is defect minimizing. Let $\bar{\phi}$ be the associated homogeneous quasimorphism. Then we have $\|\bar{\phi} - \phi\| \leq D(\phi)$ and $D(\bar{\phi}) \leq 2D(\phi)$; see Proposition 3.1. From the proof of [BH11, Proposition 2.2] we see that $\|\tilde{\phi} - \bar{\phi}\| \leq D(\bar{\phi})$. Combining these results we conclude that

$$\|\tilde{\phi} - \phi\| \leq 3D(\phi).$$

□

For what follows we will need to construct bi-harmonic quasimorphisms for degenerate measures. Those will still be defect minimizing but not necessarily unique. We will need the following result to show Theorem 3.6.

Lemma 3.5. *Let G be a finitely generated group and let μ be a symmetric measure on G with finite first moment. Let ϕ be a quasimorphism. Then there is an equivalent quasimorphism $\tilde{\phi}$ which satisfies $\|\phi - \tilde{\phi}\| \leq 3D(\phi)$ such that $\tilde{\phi}$ is bi-harmonic with respect to μ and defect minimizing among all quasimorphisms equivalent to it.*

Proof. Fix a finite generating set S of G. Let $\tilde{S} = S \setminus \operatorname{supp}(\mu)$. If $\tilde{S}$ is empty then μ is non-degenerate and we may apply Theorem 3.4. Otherwise, let ν be the uniform measure on the finite set $\tilde{S}$ and define $(\mu_n)_{n \in \mathbb{N}}$, a sequence of measures on G via $\mu_n := \frac{1}{n}\nu + \frac{n-1}{n}\mu$. It is clear that μ_n is non-degenerate and has finite first moment. Let ϕ_n be the bi-harmonic quasimorphism associated to μ_n from Theorem 3.4. We have that $\|\phi_n - \phi\| \leq 3D(\phi)$ and that every ϕ_n is defect minimizing.

Let ω be an ultrafilter on $\mathbb{N}$ and define

$$\tilde{\phi}(g) = \lim_{\omega} \phi(g).$$

By the properties of the ultrafilter it is clear that ϕ is bi-harmonic with respect to μ and defect minimizing. □

3.3. **Quasimorphisms and amenable subgroups.** For a given quasimorphism $\phi\colon G \to \mathbb{R}$ we will construct an equivalent quasimorphism $\tilde{\phi}_H$ which behaves nicely with respect to a certain amenable subgroup $H \leq G$.

We first recall the definition of a Følner sequence. Let G be a finitely generated group. A Følner sequence is a sequence $(\mathcal{F}_n)_{n\in\mathbb{N}}$ of subsets $\mathcal{F}_n \in G$ such that

- $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ for all $n \in \mathbb{N}$,
- $\cup_{n\in\mathbb{N}}\mathcal{F}_n = G$ and
- for every $g \in G$ we have that

$$\lim_{n\to\infty} \frac{|g\mathcal{F}_n \Delta \mathcal{F}_n|}{|\mathcal{F}_n|} = 0.$$

We call a Følner sequence *symmetric* if and only if $\mathcal{F}_n = \mathcal{F}_n^{-1}$.

It is well known that a finitely generated group is amenable if and only if there is a Følner sequence. Every group with subexponential growth has a symmetric Følner sequence by taking an increasing sequence of balls around the identity.

Theorem 3.6. *Let $\phi\colon G \to \mathbb{R}$ be a quasimorphism on a finitely generated group G and let $H \leq G$ be an amenable subgroup that admits a symmetric Følner sequence. Then there is a symmetric equivalent quasimorphism $\tilde{\phi}_H$ which is defect minimizing, restricts to $\bar{\phi}$ on H, satisfies $\|\tilde{\phi}_H - \phi\| \leq 3D(\phi)$, and such that for all $g \in G$, $h \in H$ we have*

$$\tilde{\phi}_H(gh) = \tilde{\phi}_H(g) + \tilde{\phi}_H(h) = \tilde{\phi}_H(hg).$$

Proof. Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a symmetric Følner sequence in H and let $\phi\colon G \to \mathbb{R}$ be a quasimorphism. Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of probability measures on G such that μ_n is uniformly supported on $\mathcal{F}_n$ and let ϕ_n be the associated sequence of quasimorphisms from Lemma 3.5. Hence, ϕ_n satisfies that

- ϕ_n is defect minimizing,
- $\|\phi - \phi_n\| \leq 3D(\phi)$, and
- for every $g \in G$ we have that $\phi_n(g) = \sum_{x\in H} \phi_n(gx)\mu_n(x) = \sum_{x\in H} \phi_n(xg)\mu_n(x)$.

Moreover, we note that $\phi_n(-x) = -\phi_n(x)$ for all $x \in G$ and thus $\sum_{x\in H} \phi_n(x)\mu_n(x) = 0$.

Finally, let ω be an ultrafilter on $\mathbb{N}$ and set define $\tilde{\phi}_H$ by setting

$$\tilde{\phi}_H(g) := \lim_{\omega} \phi_n(g).$$

We see that $\tilde{\phi}_H$ also has minimal defect and satisfies

$$\|\tilde{\phi}_H - \phi\| \leq 3D(\phi).$$

It remains to show that for all $g \in G$, $h \in H$ we have

$$\tilde{\phi}_H(gh) = \tilde{\phi}_H(g) + \tilde{\phi}_H(h) = \tilde{\phi}_H(hg).$$

Claim 3.7. *Let $g \in G$, $h \in H$ and $\delta\phi_n(g, h) = \phi_n(gh) - \phi_n(g) - \phi_n(h)$. Then there is a constant $D > 0$, independent of n, such that*

$$\left| \sum_{x\in H} \delta\phi_n(g, hx)\mu_n(x) \right| \leq D \frac{|h\mathcal{F}_n \Delta \mathcal{F}_n|}{|\mathcal{F}_n|}.$$

Proof. We know that the function $h \mapsto \delta\phi_n(g, h)$ is uniformly bounded by the defect $D(\phi_n) \leq D(\phi)$. Thus $|\sum_{x\in H}(\delta\phi_n(g, hx) - \delta\phi_n(g, x))\mu_n(x)|$ is uniformly bounded in terms of the symmetric difference of $h\mathcal{F}_n$ and $\mathcal{F}_n$, that is,

$$\left| \sum_{x\in H} (\delta\phi_n(g, hx) - \delta\phi_n(g, x))\mu_n(x) \right| \leq D(\phi) \frac{|h\mathcal{F}_n \Delta \mathcal{F}_n|}{|\mathcal{F}_n|}.$$

By harmonicity of ϕ_n we have

$$\sum_{x\in H} \delta\phi_n(g,x)\mu_n(x) = \sum_{x\in H} \left(\phi_n(gx) - \phi_n(g) - \phi_n(x)\right)\mu_n(x) = 0,$$

from which we conclude that

$$\left|\sum_{x\in H} \delta\phi_n(g,hx)\mu_n(x)\right| \leq D(\phi)\frac{|h\mathcal{F}_n \Delta \mathcal{F}_n|}{|\mathcal{F}_n|}.$$

□

Claim 3.8. *Fix $g \in G$ and $h \in H$. There is a $D > 0$ such that*

$$|\phi_n(gh) - \phi_n(g) - \phi_n(h)| \leq D\frac{|h\mathcal{F}_n \Delta \mathcal{F}_n|}{|\mathcal{F}_n|}.$$

Proof. Observe that

$$\phi_n(gh) = \sum_{x\in H} \phi_n(ghx)\mu_n(x) = \sum_{x\in H} \left(\phi_n(g) + \phi_n(hx) + \delta\phi_n(g,hx)\right)\mu_n(x).$$

As ϕ_n is harmonic and μ_n is symmetric, we see that $\sum_{x\in H} \phi_n(hx)\mu_n(x) = \phi_n(h)$. Thus

$$\phi_n(gh) = \phi_n(g) + \phi_n(h) + \sum_{x\in H} \delta\phi_n(g,hx)\mu_n(x).$$

We conclude by the previous claim. □

Finally we can prove Theorem 3.6. By Claim 3.8 we see that $\tilde{\phi}_H(gh) - \tilde{\phi}_H(g) - \tilde{\phi}_H(h) = 0$. By the same argument we also see that $\tilde{\phi}_H(hg) - \tilde{\phi}_H(g) - \tilde{\phi}_H(h) = 0$. □

Corollary 3.9. *Let G be a finitely generated group and let $g_0 \in G$ be an element. There is a quasimorphism $\phi\colon G \to \mathbb{R}$ such that*

(i) $\bar{\phi}$ is extremal for g_0,
(ii) The defect of ϕ satisfies $2D(\phi) = D(\bar{\phi})$ and
(iii) for all $g \in G$

$$\phi(g_0 g) = \phi(g_0) + \phi(g) = \phi(g g_0).$$

Proof. Let ϕ' be any extremal quasimorphism for g_0 as in Theorem 3.3 and set $\phi = \tilde{\phi}'_{\langle g_0\rangle}$ as in Theorem 3.6. □

4. Circle quasimorphisms

In light of Bavard's Duality Theorem (Theorem 3.3), the scl-gap results of $\frac{1}{2}$ must be realized by quasimorphisms. Any group G with a strong scl-gap of $\frac{1}{2}$ satisfies that for every non-torsion $g \in G$ there is a homogeneous quasimorphism $\phi_g\colon G \to \mathbb{R}$ with $\phi_g(g) \geq 1$ and $D(\phi_g) = 1$. However, it is notoriously difficult to explicitly construct these maps.

In [Heu19], the second author constructed quasimorphisms detecting strong scl gaps of 1/2 for some classes of group; see Subsection 2.1. Those quasimorphisms are induced by an action on the circle, though this action was not made explicit. In this section we will give explicit constructions of such actions and generalize the key theorem in [Heu19].

We briefly recall the correspondence between circle actions and quasimorphisms. See Subsection 2.4 for details. Let

$$\mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}) = \{\phi \mid \phi\colon \mathbb{R} \to \mathbb{R} \text{ orientation-preserving homeomorphism}, \phi \circ T = T \circ \phi\}$$

be the group of orientation-preserving homeomorphisms of $\mathbb{R}$ which commute with the the integer shift $T\colon \mathbb{R} \to \mathbb{R}$ with $T\colon x \to x+1$. For what follows we will work with $\mathrm{Homeo}^+_{M\mathbb{Z}}(\mathbb{R})$ which is analogous to the definition of $\mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$ except that we require ϕ to commute with shifts by an integer M (i.e. with T^M) instead.

The *rotation number* $\mathrm{rot}\colon \mathrm{Homeo}^+_{M\mathbb{Z}}(\mathbb{R}) \to \mathbb{R}$ is defined as

$$\mathrm{rot}\colon \phi \mapsto \lim_{n\to\infty} \frac{\phi^n(0)}{n}$$

and can be seen to be a homogeneous quasimorphism on $\mathrm{Homeo}_{M\mathbb{Z}}(\mathbb{R})$ of defect M. For any homomorphism $\rho\colon G \to \mathrm{Homeo}_{M\mathbb{Z}}(\mathbb{R})$ we may pull back rot to a homogeneous quasimorphism $\rho^*\mathrm{rot}$ on G of defect at most M. Those will be the quasimorphisms arising in this section.

In Subsection 4.1, we give an explicit and easy construction of such quasimorphisms for the non-abelian free group $F(\{\mathtt{a},\mathtt{b}\})$ on two generators. Given an alternating element $b \in \mathcal{A}$ of even length we will construct a map $\bar{\rho}_b\colon F(\{\mathtt{a},\mathtt{b}\}) \to \mathrm{Maps}(\mathbb{Z} \to \mathbb{Z})$ and associate a rotation number $\bar{\mathrm{rot}}\colon \mathrm{Maps}(\mathbb{Z} \to \mathbb{Z}) \to \mathbb{R}$ such that $\bar{\rho}_b^*\bar{\mathrm{rot}}$ is a homogeneous quasimorphism on $F(\{\mathtt{a},\mathtt{b}\})$ of defect $|b|$; see Theorem 4.4. The map $\bar{\rho}_b$, however, is not an honest homomorphism, as maps $\mathbb{Z} \to \mathbb{Z}$ may not be invertible. In Subsections 4.3 and 4.4 we will define circle words $\tilde{\mathcal{C}}_M$, the group $\mathrm{Homeo}^+_{M\mathbb{Z}}(\tilde{\mathcal{C}}_M)$ and a map $\mathrm{rot}_M : \mathrm{Homeo}^+_{M\mathbb{Z}}(\tilde{\mathcal{C}}_M) \to \mathbb{R}$ which we will show to be a quasimorphism of defect M. All quasimorphisms of this section are induced by honest homomorphisms $\rho\colon G \to \mathrm{Homeo}^+_{M\mathbb{Z}}(\tilde{\mathcal{C}}_M)$. In Subsection 4.5 we see how elements in $\mathrm{Homeo}^+_{M\mathbb{Z}}(\tilde{\mathcal{C}}_M)$ may be realized by elements in $\mathrm{Homeo}^+_{M\mathbb{Z}}(\mathbb{R})$. Finally in Subsection 4.6 we will construct extremal quasimorphisms for free groups and prove Theorem 4.4.

4.1. **Circle quasimorphisms.** In this subsection we define a new type of quasimorphism $\bar{\mathrm{rot}}_b$ on $F(\{\mathtt{a},\mathtt{b}\})$, the non-abelian free groups with generators $\{\mathtt{a},\mathtt{b}\}$. These quasimorphisms will be called *circle quasimorphisms*. We will decorate all maps or quasimorphisms constructed in this subsection with a bar-symbol (e.g. $\bar{\rho}_b$, $\bar{\mathrm{rot}}_b$) to indicate that they are not honest actions or rotation numbers (e.g. ρ, rot) defined in the later subsections.

A word $w \in F(\{\mathtt{a},\mathtt{b}\})$ is called *alternating* if its letters alternate between letters of $\{\mathtt{a},\mathtt{a}^{-1}\}$ and $\{\mathtt{b},\mathtt{b}^{-1}\}$. For example $\mathtt{aba}^{-1}\mathtt{b}$ is an alternating word but $\mathtt{abaab}$ is not. We denote by $\mathcal{A}$ the set of all alternating words.

Fix a cyclically reduced word $b \in \mathcal{A}$ with $b = \mathtt{x}_0 \cdots \mathtt{x}_{M-1}$ of even length. For every letter $\mathtt{y} \in \{\mathtt{a},\mathtt{b},\mathtt{a}^{-1},\mathtt{b}^{-1}\}$ we define a function $\bar{\rho}_b(\mathtt{y})\colon \mathbb{Z} \to \mathbb{Z}$ by setting

$$\bar{\rho}_b(\mathtt{y})\colon i \mapsto \begin{cases} i+1 & \text{if } \mathtt{y} = \mathtt{x}_i \\ i-1 & \text{if } \mathtt{y} = \mathtt{x}_{i-1}^{-1} \\ i & \text{else,} \end{cases}$$

where the indices i are considered mod M. This map is well defined, as we can not have $\mathtt{y} = \mathtt{x}_i$ and $\mathtt{y} = \mathtt{x}_{i-1}^{-1}$ since in this case b would not be cyclically reduced. For every element $f \in F(\{\mathtt{a},\mathtt{b}\})$ with $f = \mathtt{y}_1 \cdots \mathtt{y}_m$ reduced we define $\bar{\rho}_b(f)\colon \mathbb{Z} \to \mathbb{Z}$ via

$$\bar{\rho}_b(f) := \bar{\rho}_b(\mathtt{y}_1) \circ \cdots \circ \bar{\rho}_b(\mathtt{y}_m)$$

We make an easy but useful observation:

Lemma 4.1. *Let $b \in \mathcal{A}$ be an alternating word of even length and let $w \in F(\{\mathtt{a},\mathtt{b}\})$ be a reduced element which may be written (reduced) as $w = w_1\mathtt{x}\mathtt{x}w_2$ where $\mathtt{x} \in \{\mathtt{a},\mathtt{b},\mathtt{a}^{-1},\mathtt{b}^{-1}\}$. Set $w' = w_1\mathtt{x}w_2$. Then $\bar{\rho}_b(w) = \bar{\rho}_b(w')$.*

Proof. Observe that for every letter $\mathtt{x}$ we have that $\bar{\rho}_b(\mathtt{xx}) = \bar{\rho}_b(\mathtt{x}) \circ \bar{\rho}_b(\mathtt{x}) = \bar{\rho}_b(\mathtt{x})$. □

The map $\bar{\rho}_b$ is *not* a homomorphism (see Example 4.3) but shares many similarities with honest homomorphisms $\rho\colon F(\{\mathtt{a},\mathtt{b}\}) \to \mathrm{Homeo}^+_{M\mathbb{Z}}(\mathbb{R})$ described in the introduction of this section and where $M = |b|$. For example, for every pair of integers $i \le j$ we have that $\bar{\rho}_b(w).i \le \bar{\rho}_b(w).j$, and $\bar{\rho}_b(w)$ is invariant under shifts by the integer M.

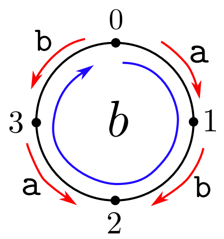

FIGURE 4. Example 4.3, visualization of $\bar{\rho}_b$: Here $b = \mathtt{a}\mathtt{b}\mathtt{a}^{-1}\mathtt{b}^{-1}$ and the circle is labeled clockwise by b (blue arrow) starting at 0. Every edge is labeled by a letter in $\{\mathtt{a}, \mathtt{b}\}$ with an orientation (red arrows).

Definition 4.2 (Circle Quasimorphisms)**.** Let $b \in \mathcal{A}$ be an alternating word of even length and let $\bar{\rho}_b \colon F(\{\mathtt{a}, \mathtt{b}\}) \to \mathrm{Maps}(\mathbb{Z} \to \mathbb{Z})$ be as above. The map $\bar{\mathrm{rot}}_b \colon F(\{\mathtt{a}, \mathtt{b}\}) \to \mathbb{R}$ defined via

$$\bar{\mathrm{rot}}_b \colon w \mapsto \lim_{n \to \infty} \frac{\bar{\rho}_b(w^n).0}{n},$$

is called a *circle quasimorphism with base b.*

We will see in Theorem 4.4 that circle quasimorphisms are indeed homogeneous quasimorphisms of defect at most $|b|$ and that they are induced by an honest action on a circle.

Example 4.3. We think of the map $\bar{\rho}_b$ mod M as walking a long a circle labeled by the base b. We illustrate this in Figure 4, where $b = \mathtt{a}\mathtt{b}\mathtt{a}^{-1}\mathtt{b}^{-1}$ and $M = 4$. The base b labels the edges of the circle clockwise starting at 0. Here, $\bar{\rho}_b(\mathtt{a})$ may be described as follows. For every $i \in \{0, 1, 2, 3\}$ we "follow an arrow labeled by $\mathtt{a}$ if we can". For example, $\bar{\rho}_b(\mathtt{a}).0 = 1$ as there is an arrow labeled by $\mathtt{a}$ from 0 to 1 and $\bar{\rho}_b(\mathtt{a}).1 = 1$ as the $\mathtt{a}$-arrow goes towards 1. Also, $\bar{\rho}_b(\mathtt{a}).3 = 2$ as there is an $\mathtt{a}$-arrow from 3 to 2. We define $\bar{\rho}_b(\mathtt{a}^{-1})$ by following the arrow labeled by $\mathtt{a}$ backwards, for example $\bar{\rho}_b(\mathtt{a}^{-1}).2 = 3$. Define $\bar{\rho}_b(\mathtt{b})$ and $\bar{\rho}_b(\mathtt{b}^{-1})$ analogously. The map $\bar{\rho}_b$ is not a monoid homomorphism to $\mathrm{Maps}(\mathbb{Z} \to \mathbb{Z})$, for example $\bar{\rho}(\mathtt{a}^{-1})\bar{\rho}(\mathtt{a}).1 = \bar{\rho}(\mathtt{a}^{-1}).1 = 0$ but $\bar{\rho}(id).1 = 1$.

Using this geometric realization we can quickly evaluate $\bar{\rho}_b$ for any element. For example $\bar{\rho}_b(\mathtt{b}^{-1}\mathtt{a}^{-1}\mathtt{b}\mathtt{a}).0 = 4$, following the path $0 \to 1 \to 2 \to 3 \to 4$ and $\bar{\rho}_b(\mathtt{a}\mathtt{b}\mathtt{a}\mathtt{b}^{-1}).0 = 2$, following the path $0 \to 0 \to 1 \to 2 \to 2$. Similarly $\bar{\rho}_b((\mathtt{b}^{-1}\mathtt{a}^{-1}\mathtt{b}\mathtt{a})^n).0 = 4n$ and $\bar{\rho}_b((\mathtt{a}\mathtt{b}\mathtt{a}\mathtt{b}^{-1})^n).0 = 2$ for all $n \geq 1$. Thus we see that $\bar{\mathrm{rot}}_b(\mathtt{b}^{-1}\mathtt{a}^{-1}\mathtt{b}\mathtt{a}) = 4$ and $\bar{\mathrm{rot}}_b(\mathtt{a}\mathtt{b}\mathtt{a}\mathtt{b}^{-1}) = 0$.

The main result of this section is the following theorem:

Theorem 4.4. *Let $b \in \mathcal{A}$ be an element of even length. Then the associated circle quasimorphism $\bar{\mathrm{rot}}_b \colon F(\{\mathtt{a}, \mathtt{b}\}) \to \mathbb{R}$ (Definition 4.2) is a homogeneous quasimorphism of defect at most $|b|$. There is an explicit action $\rho_b \colon F(\{\mathtt{a}, \mathtt{b}\}) \to \mathrm{Homeo}^+_{|b|\mathbb{Z}}(S^1)$ such that $\bar{\mathrm{rot}}_b$ agrees with the pullback of the rotation number on $\mathrm{Homeo}^+_{|b|\mathbb{Z}}(S^1)$ i.e. such that*

$$\rho_b^* \mathrm{rot} = \bar{\mathrm{rot}}_b.$$

We postpone the proof of Theorem 4.4 to Subsection 4.6.

For what follows we will define a map $\Psi \colon F(\{\mathtt{a}, \mathtt{b}\}) \to \mathcal{A}$ by setting

$$\Psi \colon \mathtt{y}_1^{n_1} \cdots \mathtt{y}_k^{n_k} \mapsto \mathtt{y}_1^{\mathrm{sign}(n_1)} \cdots \mathtt{y}_k^{\mathrm{sign}(n_k)},$$

where the $\mathtt{y}_i$ alternate between $\mathtt{a}$ and $\mathtt{b}$. For an alternating element $x = \mathtt{y}_1 \cdots \mathtt{y}_k \in \mathcal{A}$ denote by $\bar{x} := \mathtt{y}_k \cdots \mathtt{y}_1$ the *reverse of x* and observe that $\bar{x} \in \mathcal{A}$.

Proposition 4.5. *Circle quasimorphisms realize the* scl *gap of $\frac{1}{2}$ on $F(\{\mathtt{a}, \mathtt{b}\})$.*

Proof. Let $w \in F(\{\mathtt{a}, \mathtt{b}\})$ be an element in the commutator subgroup. If necessary, replace w by a conjugate which is cyclically reduced and such that w starts with a power of $\mathtt{a}$ and ends in a power of $\mathtt{b}$. Suppose that $w = \mathtt{y}_1^{n_1} \cdots \mathtt{y}_k^{n_k}$ where $\mathtt{y}_i$ alternate between $\mathtt{a}$ and $\mathtt{b}$. Let $b = \Psi(w)$ and observe that b is alternating and of even length, as b starts with $\mathtt{a}^{\pm 1}$ and ends with $\mathtt{b}^{\pm 1}$. Let $\bar{b}$ be the reverse of b. Explicitly, we see that $\bar{b} = \mathtt{y}_k^{\mathrm{sign}(n_k)} \cdots \mathtt{y}_1^{\mathrm{sign}(n_1)}$. Then by Lemma 4.1 we see that $\bar{\rho}_{\bar{b}}(w) = \bar{\rho}_{\bar{b}}(\mathtt{y}_1^{\mathrm{sign}(n_1)} \cdots \mathtt{y}_k^{\mathrm{sign}(n_k)})$. Thus

$$\begin{aligned} \bar{\rho}_{\bar{b}}(w).0 &= \bar{\rho}_{\bar{b}}(\mathtt{y}_1^{\mathrm{sign}(n_1)} \cdots \mathtt{y}_k^{\mathrm{sign}(n_k)}).0 \\ &= \bar{\rho}_{\bar{b}}(\mathtt{y}_1^{\mathrm{sign}(n_1)} \cdots \mathtt{y}_{k-1}^{\mathrm{sign}(n_{k-1})}).1 \\ &\cdots \\ &= \bar{\rho}_{\bar{b}}(\mathtt{y}_1^{\mathrm{sign}(n_1)}).(k-1) \\ &= k, \end{aligned}$$

and similarly $\bar{\rho}_{\bar{b}}(w^n) = k \cdot n$. Thus $\bar{\mathrm{rot}}_{|\bar{b}|}(w) = k = |\bar{b}|$, and by Bavard's duality (Theorem 3.3),

$$\mathrm{scl}_{F(\{\mathtt{a},\mathtt{b}\})}(w) \geq \frac{\bar{\mathrm{rot}}_{\bar{b}}(w)}{2D(\bar{\mathrm{rot}}_{\bar{b}})} \geq \frac{1}{2},$$

using $D(\bar{\mathrm{rot}}_{\bar{b}}) \leq |\bar{b}|$ from Theorem 4.4. $\square$

Example 4.6. Let $w = [\mathtt{b}^{-2}, \mathtt{a}^{-1}] = \mathtt{b}^{-2}\mathtt{a}^{-1}\mathtt{b}^2\mathtt{a}$. Then set $\bar{b} = \mathtt{a}\mathtt{b}\mathtt{a}^{-1}\mathtt{b}^{-1}$ as in Example 4.3. Then the above computations show that $\bar{\mathrm{rot}}_{\bar{b}}(w) = 4$ and that $\bar{\mathrm{rot}}_{\bar{b}}$ realizes the scl-gap for w. As w is a commutator it is well known that $\mathrm{scl}_{F(\{\mathtt{a},\mathtt{b}\})}(w) = 1/2$. Thus $\bar{\mathrm{rot}}_{\bar{b}}$ is extremal for w.

We now define *letter-quasimorphisms*. These are maps $\Phi \colon G \to F(\{\mathtt{a}, \mathtt{b}\})$ which have at most one letter "as a defect". This definition is slightly more general than the original definition of [Heu19], where these maps were required to have image in $\mathcal{A}$.

Definition 4.7 (Letter Quasimorphism)**.** A *letter-quasimorphism* is a map $\Phi \colon G \to F(\{\mathtt{a}, \mathtt{b}\})$ which is alternating, i.e. $\Phi(g^{-1}) = \Phi(g)^{-1}$ for all $g \in G$, and such that for every $g, h \in G$, one of the following cases hold:

1. $\Phi(g) \cdot \Phi(h) = \Phi(gh)$, or
2. there are elements $c_1, c_2, c_3 \in \mathcal{A}$ and a letter $\mathtt{x} \in \{\mathtt{a}, \mathtt{a}^{-1}, \mathtt{b}, \mathtt{b}^{-1}\}$ such that up to a cyclic permutation of $\Phi(g), \Phi(h), \Phi(gh)^{-1}$ we have that
$$\begin{aligned} \Phi(g) &= c_1^{-1}\mathtt{x}c_2 \\ \Phi(h) &= c_2^{-1}\mathtt{x}c_3 \\ \Phi(gh)^{-1} &= c_3^{-1}\mathtt{x}^{-1}c_1 \end{aligned}$$
where all expressions are supposed to be reduced.

An example of a letter-quasimorphism is $\Psi \colon F(\{\mathtt{a}, \mathtt{b}\}) \to \mathcal{A}$ defined above. The following Theorem 4.8 is a key result of [Heu19] stated in slightly greater generality. We now prove it using an explicit and completely different argument.

Theorem 4.8 ([Heu19, Theorem 4.7])**.** *Let $\Phi \colon G \to F(\{\mathtt{a}, \mathtt{b}\})$ be a letter-quasimorphism and let $g_0 \in G$ be an element such that there is a $K > 0$ with $\Phi(g_0^n) = b_l b_0^{n-K} b_r$ for all $n \geq K$, with $b \in F(\{\mathtt{a}, \mathtt{b}\})$ nontrivial and neither a power of $\mathtt{a}$ or $\mathtt{b}$ and where $b_l, b_r \in F(\{\mathtt{a}, \mathtt{b}\})$. Then there is a homogeneous quasimorphism $\phi \colon G \to \mathbb{R}$ such that $\phi(g) \geq 1$ and $D(\phi) \leq 1$. In particular, $\mathrm{scl}_G(g_0) \geq 1/2$.*

Proof. Let G, Φ, $g_0 \in G$ and b_0 be as in the theorem. We may assume that b_0 starts in a power of $\mathtt{a}$ and ends in a power of $\mathtt{b}$ by possibly changing b_l and b_r. Suppose that $b_0 = \mathtt{a}^{n_0}\mathtt{b}^{n_1} \ldots \mathtt{a}^{n_{M-2}}\mathtt{b}^{n_{M-1}}$

and let $b = \mathtt{a}^{\mathrm{sign}(n_0)}\mathtt{b}^{\mathrm{sign}(n_1)} \dots \mathtt{a}^{\mathrm{sign}(n_{M-2})}\mathtt{b}^{\mathrm{sign}(n_{M-1})}$. Observe that b has to be of even length. Let $\bar{b}$ be the reverse of b. Define $\phi = \Phi^* \bar{\mathrm{rot}}_{\bar{b}}$. A similar computation as in the proof of Proposition 4.5 shows that $\phi(g_0) = |\bar{b}|$.

Let $g, h \in G$. If $g, h, (gh)^{-1}$ are as in (1) of the definition of letter-quasimorphisms then we see that

$$|\phi(g) + \phi(h) - \phi(gh)| \leq |\bar{b}|,$$

as $\bar{\mathrm{rot}}_{\bar{b}}$ has defect $|\bar{b}|$. If $g, h, (gh)^{-1}$ are as in (2) of the definition of letter-quasimorphism then up to a cyclic permutation we may assume that

$$\begin{aligned} \Phi(g) &= c_1^{-1}\mathtt{x}c_2 \\ \Phi(h) &= c_2^{-1}\mathtt{x}c_3 \\ \Phi(gh) &= c_1^{-1}\mathtt{x}c_3 \end{aligned}$$

where $c_1, c_2, c_3 \in \mathcal{A}$, $\mathtt{x}$ is some letter and all expressions above are reduced. We see that

$$\bar{\mathrm{rot}}_{\bar{b}}(\Phi(gh)) = \bar{\mathrm{rot}}_{\bar{b}}(c_1^{-1}\mathtt{x}c_3) = \bar{\mathrm{rot}}_{\bar{b}}(c_1^{-1}\mathtt{x}\mathtt{x}c_3) = \bar{\mathrm{rot}}_{\bar{b}}(\Phi(g)\Phi(h))$$

using Lemma 4.1. Hence

$$\begin{aligned} |\phi(g) + \phi(h) - \phi(gh)| &= \bar{\mathrm{rot}}_{\bar{b}}(\Phi(g)) + \bar{\mathrm{rot}}_{\bar{b}}(\Phi(h)) - \bar{\mathrm{rot}}_{\bar{b}}(\Phi(gh)) \\ &= \bar{\mathrm{rot}}_{\bar{b}}(\Phi(g)) + \bar{\mathrm{rot}}_{\bar{b}}(\Phi(h)) - \bar{\mathrm{rot}}_{\bar{b}}(\Phi(g)\Phi(h)) \\ &\leq |\bar{b}| \end{aligned}$$

by Theorem 4.4. Thus $D(\phi) \leq |\bar{b}|$ and $\mathrm{scl}_G(g_0) \geq \frac{1}{2}$. $\square$

One may wonder if we can prove Theorem 4.4 in greater generality, such as for free groups on more than two generators. The following example shows that this is not the case.

Example 4.9. Theorem 4.4 just works for alternating words on 2-generator free groups as the following example shows. Consider an analogous construction of $\bar{\mathrm{rot}}_b$ for $b = \mathtt{ab}$ and for the free group $F(\{\mathtt{a}, \mathtt{b}, \mathtt{c}\})$ on three letters. Then for $g = \mathtt{baca}^{-1}$ and $h = \mathtt{ac}^{-1}\mathtt{a}^{-1}\mathtt{b}$, we see that $gh = \mathtt{bb}$. We compute that $\mathrm{rot}_b(g) = 2$, $\mathrm{rot}_b(h) = 2$ and $\mathrm{rot}_b(gh) = 0$. Thus the defect of rot_b is at least 4 but $|b| = 2$.

4.2. **Geometric proof for the strong scl-gap on free groups.** We will give a geometric proof for the strong scl-gap on free groups. We will later formalize this construction by introducing *circle words* in the next subsection. Let $g \in F(\mathcal{S})'$ be an element in the commutator subgroup of a non-abelian free group on the set $\mathcal{S}$. We will describe an explicit homomorphism $\rho \colon F(\mathcal{S}) \to \mathrm{Homeo}^+_{M\mathbb{Z}}(\mathbb{R})$ such that $\rho^*\mathrm{rot}$, the pullback of rot via ρ, is a quasimorphism which realizes the gap of $1/2$ for g.

Fix an element $g \in F(\mathcal{S})$ in the commutator subgroup of $F(\mathcal{S})$. By possibly conjugating g we may assume that $g = \mathtt{s}_0^{n_0} \cdots \mathtt{s}_{M-1}^{n_{M-1}}$ with $\mathtt{s}_i \in \mathcal{S}$ and $n_i \in \mathbb{Z}$ such that g is cyclically reduced, i.e. $\mathtt{s}_i \neq \mathtt{s}_{i+1}$ for $i \in \{0, \cdots, M-2\}$ and $\mathtt{s}_{M-1} \neq \mathtt{s}_0$. Observe that every element in the commutator subgroup of $F(\mathcal{S})$ may be conjugated to have such a form.

For the integer M above, let $\mathcal{X}_M := \{\mathtt{x}_0, \dots, \mathtt{x}_{M-1}\}$ be an alphabet independent of $\mathcal{S}$. We will define $\rho(g)$ as a product of maps $\Psi_{\mathtt{x}} \in \mathrm{Homeo}^+_{M\mathbb{Z}}(\mathbb{R})$ which we now describe, where $\mathtt{x} \in \mathcal{X}_M^{\pm}$.

For every $\mathtt{x}_i \in \mathcal{X}_M$ we define the map $\Psi_{\mathtt{x}_i}$ to be the identity *outside*

$$\bigsqcup_{j \equiv i \mod M} (j - \frac{1}{2}, j + \frac{3}{2})$$

and such that

- for all $n \geq 1$, $\Psi^n_{\mathtt{x}_i}([j, j+1]) \subset (j+1, j+\frac{3}{2})$ for all $j \equiv i \mod M$ and
- for all $n \leq -1$, $\Psi^n_{\mathtt{x}_i}([j, j+1]) \subset (j - \frac{1}{2}, j)$ for all $j \equiv i \mod M$.

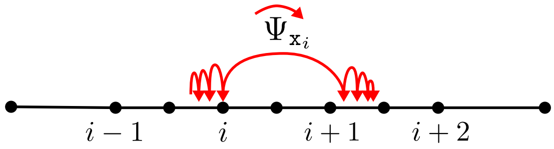

FIGURE 5. Visualization $\Psi_{\mathtt{x}_i}$: The red arrows indicate the orbits of $\Psi_{\mathtt{x}_i}^n.i$. We require that $\Psi_{\mathtt{x}_i}.i > i+1$, that $\lim_{n\to\infty} \Psi_{\mathtt{x}_i}.i \leq i + \frac{3}{2}$ and that $\lim_{n\to-\infty} \Psi_{\mathtt{x}_i}.i \geq i - \frac{1}{2}$.

See Figure 5 for a possible realization of such a map $\Psi_{\mathtt{x}_i}$. We set $\Psi_{\mathtt{x}_i^{-1}} = \Psi_{\mathtt{x}_i}^{-1}$. This defines $\Psi_{\mathtt{x}}$ for all elements $\mathtt{x} \in \mathcal{X}_M^{\pm}$. We call two elements $\mathtt{x}, \mathtt{y} \in \mathcal{X}_M^{\pm}$ *adjacent* if one of them is $\mathtt{x}_i$ or $\mathtt{x}_i^{-1}$ and the other one is $\mathtt{x}_{i+1}$ or $\mathtt{x}_{i+1}^{-1}$, where $i \in \{0, \dots, M-1\}$ and indices are taken mod M. Observe that if $\mathtt{x}, \mathtt{y}$ are not adjacent, then $\Psi_{\mathtt{x}}$ and $\Psi_{\mathtt{y}}$ commute as they have disjoint support.

We now describe a homomorphism $\rho\colon F(\mathcal{S}) \to \mathrm{Homeo}^+_{M\mathbb{Z}}(\mathbb{R})$ such that the the induced rotation quasimorphism $\rho^*\mathrm{rot}\colon F(\mathcal{S}) \to \mathbb{R}$ realizes the scl-gap of $1/2$ for g. If $\mathtt{s} \in \mathcal{S}$ is a letter which does not occur in g then we set $\rho(\mathtt{s})\colon \mathbb{R} \to \mathbb{R}$ to be the identity. If $\mathtt{s}$ occurs in g then define

$$\rho(\mathtt{s}) := \prod_{i\colon\ \mathtt{s}_i = \mathtt{s}, n_i > 0} \Psi_{\mathtt{x}_{M-1-i}} \circ \prod_{i\colon\ \mathtt{s}_i = \mathtt{s}, n_i < 0} \Psi_{\mathtt{x}^{-1}_{M-1-i}}$$

Observe that none of the letters involved in this definition are adjacent and thus all of the $\Psi_{\mathtt{x}}$ terms commute.

Extend ρ to a homomorphism $\rho\colon F(\mathcal{S}) \to \mathrm{Homeo}^+_{M\mathbb{Z}}(\mathbb{R})$.

We now evaluate $\rho(g)$. Note that

$$\rho(g) = \rho(\mathtt{s}_0^{n_0} \cdots \mathtt{s}_{M-1}^{n_{M-1}}) = \rho(\mathtt{s}_0^{\mathrm{sign}(n_0)})^{|n_0|} \circ \cdots \circ \rho(\mathtt{s}_{M-1}^{\mathrm{sign}(n_{M-1})})^{|n_{M-1}|}.$$

We first evaluate

$$\rho(s_{M-1}^{n_{M-1}}).[0,1/2) = \rho(\mathtt{s}_{M-1}^{\mathrm{sign}(n_{M-1})})^{|n_{M-1}|}.[0,1/2).$$

We see that the only Ψ in the definition of $\rho(\mathtt{s})$ which is not the identity on $[0,1/2)$ is $\Psi_{\mathtt{x}_0}$. Thus by the choice of $\Psi_{\mathtt{x}_0}$ we have

$$\rho(\mathtt{s}_{M-1}^{\mathrm{sign}(n_{M-1})})^{|n_{M-1}|}.[0,1/2) \subset (1,3/2).$$

By the same argument we see that we have that

$$\rho(\mathtt{s}_{M-2}^{\mathrm{sign}(n_{M-2})})^{|n_{M-2}|}.(1,3/2) \subset (2,5/2).$$

By induction we see that $\rho(g).0 \in (M, M+1/2)$, and that $\rho(g^n).0 \in (nM, nM+1/2)$. Thus $\rho^*\mathrm{rot}(g) = M$. By Theorem 2.14 we have that $D(\rho^*\mathrm{rot}(g)) \leq M$. Thus $\rho^*\mathrm{rot}(g)$ realizes the scl gap of $1/2$ for g.

We summarize these results in the following theorem.

Theorem 4.10. *Let $F(\mathcal{S})$ be the a free group on the set $\mathcal{S}$ and let $g \in F(\mathcal{S})$ be an element in the commutator subgroup. There exist an explicit homomorphism $\rho\colon F(\mathcal{S}) \to \mathrm{Homeo}^+_{M\mathbb{Z}}(\mathbb{R})$ such that $\rho^*\mathrm{rot}$, the pullback of the rotation number via ρ, is a homogeneous quasimorphism that realizes the* scl*-gap of $1/2$ for g.*

Example 4.11. Let $g = [\mathtt{b}^{-2}, \mathtt{a}^{-1}] \in F(\{\mathtt{a}, \mathtt{b}\})$. We will describe an explicit map $\rho\colon F(\{\mathtt{a},\mathtt{b}\}) \to \mathrm{Homeo}^+_{4\mathbb{Z}}$ which realizes the scl-gap of $1/2$ for g. Indeed, let $M = 4$ and set

$$\begin{aligned} \rho(\mathtt{a}) &= \Psi_{\mathtt{x}_1} \circ \Psi_{\mathtt{x}_3^{-1}}, \text{ and} \\ \rho(\mathtt{b}) &= \Psi_{\mathtt{x}_2} \circ \Psi_{\mathtt{x}_4^{-1}}. \end{aligned}$$

Extend ρ to a homomorphism. Then the above computations show that $\rho^*\mathrm{rot}$ is an extremal quasimorphism for g.

4.3. **Circle words.** In what follows we will describe for each integer $M \geq 2$ a totally ordered set $(\tilde{\mathcal{C}}_M, \prec)$ called *circle words* and a function $\lambda \colon \tilde{\mathcal{C}}_M \to \mathbb{Z}$ respecting the order of these circle words. In Subsection 4.5 we will give a geometric interpretation of this set as a subset of $\mathbb{R}$. We think of this ordered set as a subset of the real line and where $\lambda(x) \in \mathbb{Z}$ is the closest point projection to $\mathbb{Z}$.

Fix an integer $M \geq 2$ and let $\mathtt{x}_0, \cdots, \mathtt{x}_{M-1}$ be distinct letters. We will define sets $\{\mathcal{C}_M^k\}_{k\in\mathbb{N}}$ of certain reduced words in $\{\mathtt{x}_0^{\pm}, \ldots, \mathtt{x}_{M-1}^{\pm}\}$ inductively with the property that any proper suffix of a word $v \in \mathcal{C}_M^k$ lies in $\mathcal{C}_M^{k-1}$ for all $k \geq 1$. Set $\mathcal{C}_M^0 = \{\emptyset\}$ and define λ on $\mathcal{C}_M^0$ by setting $\lambda(\emptyset) = 0$. Suppose that $\mathcal{C}_M^{k-1}$ and $\lambda \colon \mathcal{C}_M^{k-1} \to \mathbb{Z}$ are defined for some $k \geq 1$ satisfying the desired property. For any $w \in \mathcal{C}_M^{k-1}$ with $\lambda(w) \equiv i \mod M$, set

$$\mathcal{C}_M(w)_0 := \{\mathtt{x}_i w, \mathtt{x}_i^{-1} w, w, \mathtt{x}_{i-1} w, \mathtt{x}_{i-1}^{-1} w\}$$

and let $\mathcal{C}_M(w)$ be the *reduced* representatives of words in $\mathcal{C}_M(w)_0$. If $w' \in \mathcal{C}_M(w)$ is w or represents a non-reduced word in $\mathcal{C}_M(w)_0$, then $w' \in \mathcal{C}_M^{k-1}$ and thus $\lambda(w')$ has been defined. Otherwise, we have a reduced word $w' = \mathtt{y}w$ and define

$$\lambda(w') := \begin{cases} \lambda(w) + 1 & \text{if } \mathtt{y} = \mathtt{x}_i \\ \lambda(w) & \text{if } \mathtt{y} = \mathtt{x}_i^{-1} \text{ or } \mathtt{y} = \mathtt{x}_{i-1}, \text{ and} \\ \lambda(w) - 1 & \text{if } \mathtt{y} = \mathtt{x}_{i-1}^{-1}. \end{cases}$$

Finally we define

$$\begin{aligned} \mathcal{C}_M^k &:= \bigcup_{w\in\mathcal{C}_M^{k-1}} \mathcal{C}(w), & \tilde{\mathcal{C}}_M^k &:= \mathcal{C}_M^k \times \mathbb{Z} \\ \mathcal{C}_M &:= \cup_{k\in\mathbb{N}} \mathcal{C}_M^k, & \tilde{\mathcal{C}}_M &:= \mathcal{C}_M \times \mathbb{Z}. \end{aligned}$$

We call $\tilde{\mathcal{C}}_M$ *circle words.* Then λ naturally extends to $\mathcal{C}_M$. We further extend λ to a function on $\tilde{\mathcal{C}}_M$ by setting $\lambda(w, n) = \lambda(w) + nM$.

Example 4.12. Let $M = 3$. Then $\mathcal{C}_3^0 = \{\emptyset\}$, $\mathcal{C}_3^1 = \{\mathtt{x}_2^{-1}, \mathtt{x}_2, \emptyset, \mathtt{x}_0, \mathtt{x}_0^{-1}\}$ and, for example, $\mathtt{x}_2\mathtt{x}_1\mathtt{x}_0\mathtt{x}_2\mathtt{x}_1\mathtt{x}_0 \in \mathcal{C}_3^6$.

We observe the following by induction:

Claim 4.13. *If $w \neq \emptyset$ and $\lambda(w) \equiv i \mod M$ then the first letter of w is either $\mathtt{x}_{i-1}$ or $\mathtt{x}_i^{-1}$.*

We define $\prec$ on $\tilde{\mathcal{C}}_M$ inductively. On $\tilde{\mathcal{C}}_M^0 = \{(\emptyset, n) \mid n \in \mathbb{Z}\}$ we define $(\emptyset, n) \prec (\emptyset, m)$ if $n < m$ and $(\emptyset, n) \succ (\emptyset, m)$ if $n > m$.

Suppose that $\prec$ has been defined on $\tilde{\mathcal{C}}_M^{k-1}$ for some $k \geq 1$ and let $(v, n), (w, m) \in \tilde{\mathcal{C}}_M^k$ be two elements. Then

- set $(v, n) \prec (w, m)$, if $\lambda(v, n) < \lambda(w, m)$ and
- set $(v, n) \succ (w, m)$, if $\lambda(v, n) > \lambda(w, m)$.
- Otherwise we have that $\lambda(v, n) = \lambda(w, m)$ and in this case let $\lambda(v, n) = \lambda(w, m) \equiv i \mod M$. Moreover, let $\mathtt{y}_v$ be the first letter of v and let $\mathtt{y}_w$ be the first letter of w such that $v = \mathtt{y}_v v'$ and $w = \mathtt{y}_w w'$. By Claim 4.13 we have that $\mathtt{y}_v, \mathtt{y}_w \in \{\mathtt{x}_{i-1}, \mathtt{x}_i^{-1}\}$. Then
  - if $\mathtt{y}_v = \mathtt{x}_i^{-1}$ and $\mathtt{y}_w = \mathtt{x}_{i-1}$ we set $(v, n) \prec (w, m)$,
  - if $\mathtt{y}_v = \mathtt{x}_{i-1}$ and $\mathtt{y}_w = \mathtt{x}_i^{-1}$ we set $(v, n) \succ (w, m)$.
  - Otherwise, we have that $\mathtt{y}_v = \mathtt{y}_w$. Observe that $(v', n), (w', n) \in \tilde{\mathcal{C}}_M^{k-1}$ and thus the order between them is defined. We set $(v, n) \prec (w, m)$ if $(v', n) \prec (w', m)$ and $(v, n) \succ (w, m)$ if $(v', n) \succ (w', m)$.

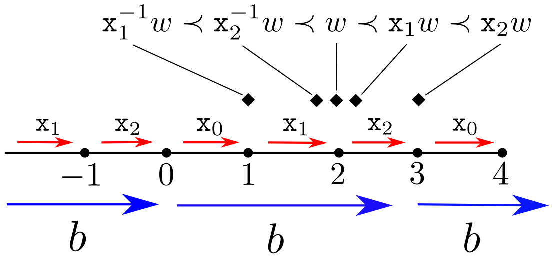

FIGURE 6. Example 4.14

This completes the definition of $(\tilde{\mathcal{C}}_M, \prec)$.

Example 4.14. An element $w \in \mathcal{C}_M$ can be realized as paths along the integer line, labeled by $b = \mathtt{x}_0 \cdots \mathtt{x}_{M-1}$. The final position of this paths is recorded as $\lambda(w)$. If $w \in \mathcal{C}_M$ has $\lambda(w) \equiv i \mod M$, then we are positioned between arrows labeled by $\mathtt{x}_i$ and $\mathtt{x}_{i-1}$. Consider the case where $M = 3$ (see Figure 6). Let $w = \mathtt{x}_1\mathtt{x}_0$. Then $\lambda(w) = 2$. Now we can go one step right via $\mathtt{x}_2$ s.t. $\lambda(\mathtt{x}_2 w) = 3$, or one step left via $\mathtt{x}_1^{-1}$ to $\mathtt{x}_1^{-1}w = \mathtt{x}_0$, hence $\lambda(\mathtt{x}_1^{-1}w) = 1$. We could also go a very small amount to the right via $\mathtt{x}_1$ or a very small amount to the left via $\mathtt{x}_2^{-1}$.

Similarly, elements $(w, n) \in \tilde{\mathcal{C}}_M$ correspond to such paths for $w \in \mathcal{C}_M$ shifted by nM.

4.4. **The group** $\mathrm{Homeo}^+_{M\mathbb{Z}}(\tilde{\mathcal{C}}_M)$**.** We define a group $\mathrm{Homeo}^+_{M\mathbb{Z}}(\tilde{\mathcal{C}}_M)$ analogous to the group $\mathrm{Homeo}^+_{M\mathbb{Z}}(\mathbb{R})$ (see Subsection 2.4). First let $\tau_M : \tilde{\mathcal{C}}_M \to \tilde{\mathcal{C}}_M$ be the map $\tau_M : (w, n) \mapsto (w, n+1)$. We set

$$\mathrm{Homeo}^+_{M\mathbb{Z}}(\tilde{\mathcal{C}}_M) = \{\phi \colon \tilde{\mathcal{C}}_M \to \tilde{\mathcal{C}}_M \mid \phi \text{ invertible, orientation-preserving, } \phi \circ \tau_M = \tau_M \circ \phi\}$$

Define the *rotation number* $\mathrm{rot}_M \colon \mathrm{Homeo}^+_{M\mathbb{Z}}(\tilde{\mathcal{C}}_M) \to \mathbb{R}$ by setting

$$\mathrm{rot}_M \colon \psi \mapsto \lim_{n\to\infty} \frac{\lambda(\psi^n.(\emptyset, 0))}{n}.$$

Analogous to Theorem 2.14 we see:

Theorem 4.15. *The map* rot_M *is a homogeneous quasimorphism of defect* M*.*

Proof. The proof is exactly analogous to [Cal09b, Lemmas 2.40, 2.41], which in particular implies that the limit in the definition of rot_M always exists. □

We now define elements $\psi_{\mathtt{x}} \in \mathrm{Homeo}^+_{M\mathbb{Z}}(\tilde{\mathcal{C}}_M)$ for every $\mathtt{x} \in \{\mathtt{x}_0^{\pm}, \ldots, \mathtt{x}_{M-1}^{\pm}\}$. Let

$$\psi_{\mathtt{x}_i} \colon (w, n) \mapsto \begin{cases} (\mathtt{x}_i w, n) & \text{if } \lambda(w, n) \equiv i, i+1 \mod M \\ (w, n) & \text{else,} \end{cases}$$

and

$$\psi_{\mathtt{x}_i^{-1}} \colon (w, n) \mapsto \begin{cases} (\mathtt{x}_i^{-1} w, n) & \text{if } \lambda(w, n) \equiv i, i+1 \mod M \\ (w, n) & \text{else,} \end{cases}$$

for all $i \in \{0, \ldots, M-1\}$, where we always make $\mathtt{x}_i w$ and $\mathtt{x}_i^{-1} w$ reduced and consider them as elements in $\mathcal{C}_M$.

We observe the following:

Lemma 4.16.

- *Suppose* $\mathtt{x} = \mathtt{x}_i$*. Then*
  - *if* $\lambda(w, n) \equiv i \mod M$*, then* $\lambda(\psi_{\mathtt{x}}(w, n)) \equiv i, i+1 \mod M$*, and*

– *if* $\lambda(w,n) \equiv i+1 \mod M$*, then* $\lambda(\psi_{\boldsymbol{x}}(w,n)) \equiv i+1 \mod M$*.*

- *Suppose* $\boldsymbol{x} = \boldsymbol{x}_i^{-1}$*. Then*
  – *if* $\lambda(w,n) \equiv i+1 \mod M$*, then* $\lambda(\psi_{\boldsymbol{x}}(w,n)) \equiv i, i+1 \mod M$*, and*
  – *if* $\lambda(w,n) \equiv i \mod M$*, then* $\lambda(\psi_{\boldsymbol{x}}(w,n)) \equiv i \mod M$*.*

The claim follows immediately from the definition of $\psi_{\mathtt{x}}$ and λ using Claim 4.13.

Proposition 4.17 (Circle Word Actions)**.** *We have that for every* $\boldsymbol{x} \in \{\boldsymbol{x}_0^{\pm}, \ldots, \boldsymbol{x}_i^{\pm}\}$*,* $\psi_{\boldsymbol{x}} \in \mathrm{Homeo}^+_{M\mathbb{Z}}(\tilde{\mathcal{C}}_M)$*. Moreover,* $\psi_{\boldsymbol{x}} \circ \psi_{\boldsymbol{x}^{-1}} = \mathrm{id}_{\tilde{\mathcal{C}}_M}$*.*

Proof. It is easy to see that $\psi_{\mathtt{x}}$ commutes with τ_M and that $\psi_{\mathtt{x}} \circ \psi_{\mathtt{x}^{-1}} = \mathrm{id}$ by Lemma 4.16.

We now show that $\psi_{\mathtt{x}}$ is order-preserving. Suppose that $\mathtt{x} = \mathtt{x}_i$. From the definition of $\psi_{\mathtt{x}}$ and λ it is evident that for every $(v,m) \in \tilde{\mathcal{C}}_M$ we have that $\lambda(\psi_{\mathtt{x}}.(v,m)) = \lambda(v,m)$ or $\lambda(\psi_{\mathtt{x}}.(v,m)) = \lambda(v,m)+1$.

Let $(v,m),(w,n) \in \tilde{\mathcal{C}}_M$ be such that $(v,m) \prec (w,n)$. By Lemma 4.16 we see that if $\lambda(v,m) < \lambda(w,n)$ then also $\lambda(\psi_{\mathtt{x}}.(v,m)) < \lambda(\psi_{\mathtt{x}}.(w,n))$, thus $\psi_{\mathtt{x}}.(v,m) \prec \psi_{\mathtt{x}}.(w,n)$. If $\lambda(v,m) = \lambda(w,n) \not\equiv i, i+1 \mod M$ then $\psi_{\mathtt{x}}$ is the identity and thus $\psi_{\mathtt{x}}.(v,m) \prec \psi_{\mathtt{x}}.(w,n)$. If $\lambda(v,m) = \lambda(w,n) \equiv i, i+1 \mod M$ then $\psi_{\mathtt{x}}$ multiplies the elements w and v by $\mathtt{x}_i$ which also preserves the order. □

4.5. **Geometric realization of $\psi_{\mathtt{x}}$ as maps $\Psi_{\mathtt{x}}$.** We may realize the combinatorial constructions of the last subsection by an explicit action using the maps $\Psi_{\mathtt{x}}$ defined in Subsection 4.2. Analogously to the functions λ and τ_M on $\tilde{\mathcal{C}}_M$ define $\Lambda\colon \mathbb{R} \to \mathbb{Z}$ via $\Lambda\colon [n-\frac{1}{2}, n+\frac{1}{2}) \mapsto n$ for all $n \in \mathbb{Z}$ and $T_M\colon \mathbb{R} \to \mathbb{R}$ via $T_M\colon x \mapsto x+M$ for every $x \in \mathbb{R}$.

We define a map $\Theta\colon \tilde{\mathcal{C}}_M \to \mathbb{R}$ as follows: For any $(w,n) \in \tilde{\mathcal{C}}_M$ with $w = \mathtt{y}_1 \cdots \mathtt{y}_m$, where $\mathtt{y}_i \in \{\mathtt{x}_0^{\pm}, \ldots, \mathtt{x}_{M-1}^{\pm}\}$, set

$$\Theta\colon (w,n) \mapsto T_M^n \circ \Psi_{\mathtt{y}_1} \circ \cdots \circ \Psi_{\mathtt{y}_m}(0).$$

Proposition 4.18. *We have that*

(1) the map $\Theta\colon (\tilde{\mathcal{C}}_M, \prec) \to (\mathbb{R}, <)$ *is order-preserving and for every* $(w,n) \in \tilde{\mathcal{C}}_M$ *we have*

$$\lambda(w,n) = \Lambda \circ \Theta(w,n).$$

Moreover, if $w \neq \emptyset$ *and the first letter of* w *is* $\boldsymbol{x}_i$ *then* $\Theta(w,n) \in (j+1, j+\frac{3}{2})$ *for some* $j \equiv i \mod M$ *and if the first letter is* $\boldsymbol{x}_i^{-1}$ *then* $\Theta(w,n) \in (j-\frac{1}{2}, j)$ *for some* $j \equiv i \mod M$*.*

(2) for every $(w,n) \in \tilde{\mathcal{C}}_M$ *we have that*
- $\Theta \circ \psi_{\boldsymbol{x}}(w,n) = \Psi_{\boldsymbol{x}} \circ \Theta(w,n)$*, for every* $\boldsymbol{x} \in \{\boldsymbol{x}_0^{\pm}, \ldots, \boldsymbol{x}_{M-1}^{\pm}\}$*, and*
- $\Theta \circ \tau_M(w,n) = T_M \circ \Theta(w,n)$*.*

(3) Let $\psi \in \mathrm{Homeo}^+_{M\mathbb{Z}}(\tilde{\mathcal{C}}_M)$ *be a composition of maps* $\psi_{\boldsymbol{x}}$ *(with* $\boldsymbol{x} \in \{\boldsymbol{x}_0^{\pm}, \ldots, \boldsymbol{x}_{M-1}^{\pm}\}$*) and* τ_M *and let* $\Psi \in \mathrm{Homeo}^+_{M\mathbb{Z}}(\mathbb{R})$ *be the corresponding product of* $\Psi_{\boldsymbol{x}}$ *and* T_M*. Then*

$$\mathrm{rot}_M(\psi) = \mathrm{rot}(\Psi).$$

Proof. All statements may be easily proven for $\tilde{\mathcal{C}}_M^k$ by induction on k. □

4.6. **Quasimorphisms realizing the scl-gap and proof of Theorem 4.4.** We can now prove Theorem 4.4.

Proof of Theorem 4.4. Let $b = \mathtt{z}_0 \cdots \mathtt{z}_{M-1} \in \mathcal{A}$ be an alternating word. For the given M, consider the alphabet $\mathcal{X}_M^{\pm} = \{\mathtt{x}_0^{\pm}, \ldots, \mathtt{x}_{M-1}^{\pm}\}$, circle words $\tilde{\mathcal{C}}_M$ and related notion defined in Subsections 4.3 and 4.4.

Define a map $\rho_b\colon F(\{\mathtt{a},\mathtt{b}\}) \to \mathrm{Homeo}^+_{M\mathbb{Z}}(\tilde{\mathcal{C}}_M)$ as follows. For $\mathtt{y} \in \{\mathtt{a},\mathtt{b}\}$ we define

$$\rho_b(\mathtt{y}) := \prod_{i\colon \mathtt{z}_i = \mathtt{y}} \psi_{\mathtt{x}_i} \circ \prod_{i\colon \mathtt{z}_i = \mathtt{y}^{-1}} \psi_{\mathtt{x}_i^{-1}}.$$

This extends to a homomorphism ρ_b on $F(\{\mathtt{a}, \mathtt{b}\})$.

Claim 4.19. *For every $w \in F(\{\mathtt{a}, \mathtt{b}\})$ we have*

(1) $\lambda(\rho_b(w).(\emptyset, 0)) = \bar{\rho}_b(w).0$

(2) If w starts with the letter $\mathtt{y}$ and $\lambda(\rho_b(w).(\emptyset, 0)) \equiv i \mod M$ then either $\mathtt{z}_{i-1} = \mathtt{y}$ or $\mathtt{z}_i = \mathtt{y}^{-1}$. In the first case, $\lambda(w).(\emptyset, 0)$ starts with $\mathtt{x}_{i-1}$ and in the second case $\rho_b(w).(\emptyset, 0)$ starts with $\mathtt{x}_i^{-1}$.

Proof. We prove these statements by induction on the word length $|w|$ of w. It is easy to check that if $|w| = 0, 1$, the statements are true.

Suppose the statements are true when $|w| \le k$ and suppose now $|w| = k+1$. Let $w = \mathtt{y}_1\mathtt{y}_2 w'$, where $\mathtt{y}_1, \mathtt{y}_2$ are the first and the second letters of w and for w' chosen accordingly. Without loss of generality we may assume that $\mathtt{y}_2 = \mathtt{a}$.

By the induction hypothesis we know that $\lambda(\rho_b(\mathtt{a}w').(\emptyset, 0)) = \bar{\rho}_b(\mathtt{a}w').0$ and let $i \in \{0, \dots, M-1\}$ be such that $\lambda(\rho_b(\mathtt{a}w').(\emptyset, 0)) \equiv \bar{\rho}_b(\mathtt{a}w').0 \equiv i \mod M$.

Because b is alternating and $F(\{\mathtt{a}, \mathtt{b}\})$ has only two generators, one of $\mathtt{z}_{i-1}, \mathtt{z}_i$ is $\mathtt{a}^{\pm 1}$ and the other is $\mathtt{b}^{\pm 1}$. Without loss of generality assume that $\mathtt{z}_{i-1} = \mathtt{a}^{\pm 1}$. Then by part (2) of the induction hypothesis we know that indeed $\mathtt{z}_{i-1} = \mathtt{a}$. Again without loss of generality we may assume that $\mathtt{z}_i = \mathtt{b}$. Thus, we know that $\rho_b(\mathtt{y}_2 w').(\emptyset, 0) = \rho_b(\mathtt{a}w').(\emptyset, 0) = (\mathtt{x}_{i-1}x, 0)$ for some appropriate x. Because w is reduced we know that that $\mathtt{y}_1 \in \{\mathtt{a}, \mathtt{b}^{-1}, \mathtt{b}\}$. We consider the following cases:

- If $\mathtt{y}_1 = \mathtt{a}$, then $\rho_b(w).(\emptyset, 0) = (\mathtt{x}_{i-1}\mathtt{x}_{i-1}x, 0)$ and $\lambda(\rho_b(w).(\emptyset, 0)) = \lambda(\mathtt{x}_{i-1}\mathtt{x}_{i-1}x, 0) = \lambda(\mathtt{x}_{i-1}x, 0) = \bar{\rho}_b(\mathtt{y}_2 w').0 = \bar{\rho}_b(w).0$. Item (2) of the claim is easy to check.
- If $\mathtt{y}_1 = \mathtt{b}$, then $\rho_b(w).(\emptyset, 0) = (\mathtt{x}_i\mathtt{x}_{i-1}x, 0)$ and $\lambda(\rho_b(w).(\emptyset, 0)) = \lambda(\mathtt{x}_i\mathtt{x}_{i-1}x, 0) = 1 + \lambda(\mathtt{x}_{i-1}x, 0) = 1 + \bar{\rho}_b(\mathtt{y}_2 w').0 = \bar{\rho}_b(w).0$. Item (2) of the claim is easy to check.
- If $\mathtt{y}_1 = \mathtt{b}^{-1}$, then $\rho_b(w).(\emptyset, 0) = (\mathtt{x}_i^{-1}\mathtt{x}_{i-1}x, 0)$ and $\lambda(\rho_b(w).(\emptyset, 0)) = \lambda(\mathtt{x}_i^{-1}\mathtt{x}_{i-1}x, 0) = \lambda(\mathtt{x}_{i-1}x, 0) = \bar{\rho}_b(\mathtt{y}_2 w').0 = \bar{\rho}_b(w).0$.

□

The claim above is unique to the free group on two generators.

Now it follows that for any $w \in F(\{\mathtt{a}, \mathtt{b}\})$, we have

$$\bar{\mathrm{rot}}_b(w) = \lim_{n\to\infty} \frac{\bar{\rho}_b(w^n).0}{n} = \lim_{n\to\infty} \frac{\lambda(\rho_b(w^n).(\emptyset, 0))}{n} = \rho_b^* \mathrm{rot}_M(w),$$

and since rot_M is a homogeneous quasimorphism of defect at most M, so is $\bar{\mathrm{rot}}_b$. □

5. Scl of hyperbolic elements

5.1. **Surfaces in graphs of groups.** In this subsection, we investigate surfaces in graphs of groups and their normal forms following [Che19], which will be used to estimate (relative) scl.

Let G be a graph of groups. With the setup in Subsection 2.3, let X be the standard realization of G. Let $\underline{g} = \{g_i, i \in I\}$ be a finite collection of *infinite-order* elements indexed by I and let $c = \sum r_i g_i$ be a rational chain with $r_i \in \mathbb{Q}_{>0}$. Let $\underline{\gamma} = \{\gamma_i, i \in I\}$ be tight loops representing elements in $\underline{g}$. Recall that the edge spaces cut the hyperbolic tight loops into arcs, and denote by A_v the collection of arcs supported in the thickened vertex space $N(X_v)$.

Let S be any aspherical (monotone) admissible surface for c. Put S in general position so that it is transverse to all edge spaces. Then the preimage F of the union of edge spaces is a collection of disjoint embedded proper arcs and loops. Up to homotopy and compression that simplifies S, every loop in F represents a nontrivial conjugacy class in some edge group.

Now cut S along F into subsurfaces. Then each component Σ is a surface (possibly with corner) supported in some thickened vertex space $N(X_v)$. Boundary components of Σ fall into two types (see Figure 7):

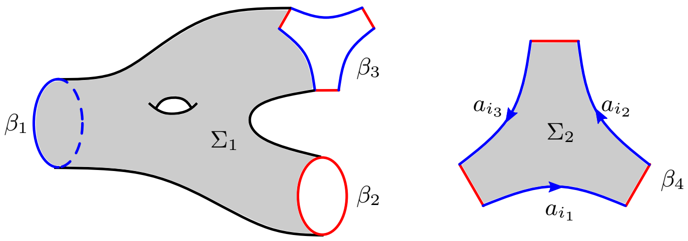

FIGURE 7. Here are two possible components Σ_1 and Σ_2 of S supported in some $N(X_v)$, where the blue parts are supported in $\underline{\gamma}$ and the red is part of F. The component Σ_1 has three boundary components: β_1 winds around an elliptic loop in $\underline{\gamma}$, the loop β_2 lies in F, and β_3 is a polygonal boundary. The component Σ_2 is a disk with polygonal boundary β_4, on which we have arcs $a_{i_1}, a_{i_2}, a_{i_3}$ in cyclic order.

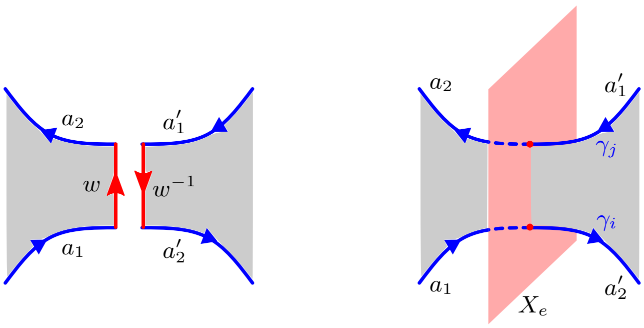

FIGURE 8. Two paired turns on an edge space X_e, with arcs $a_1, a_2' \subset \gamma_i$ and $a_1', a_2 \subset \gamma_j$

(1) *Loop boundary*: these are boundary components containing no corners. Each such boundary is either a loop in F or a loop in ∂S that winds around an elliptic tight loop in $\underline{\gamma}$;
(2) *Polygonal boundary*: these are boundary components containing corners. Each such boundary is necessarily divided into segments by the corners, such that the segments alternate between arcs in A_v and proper arcs in F (called *turns*).

Note that any component Σ with positive Euler characteristic must be a disk with polygonal boundary since elements in $\underline{g}$ have infinite-order and loops in F are nontrivial in edge spaces.

For Σ as above, we say a turn has *type* (a_1, w, a_2) for some $a_1, a_2 \in A_v$ and $w \in G_e$ for some e adjacent to v if it travels from a_1 to a_2 as a based loop supported on X_e representing w, referred to as the *winding number* of the turn. We say two turn types (a_1, w, a_2) and (a_1', w', a_2') are *paired* if they can be glued together, i.e. if there is some edge e with $w, w' \in G_e$, $a_1, a_2 \in A_{o(e)}$ and $a_1', a_2' \in A_{t(e)}$ such that $w^{-1} = w'$ and a_1 (resp. a_1') is followed by a_2' (resp. a_2) on $\underline{\gamma}$. See Figure 8.

For each vertex v, let S_v be the union of components Σ supported in $N(X_v)$ obtained from cutting S along F. Note that there is an obvious pairing of turns in $\sqcup S_v$: two turns are paired if they are glued together in S. Let $t_{(a_1,w,a_2)}$ be the number of turns of type (a_1, w, a_2) that appear in $\sqcup S_v$ *divided by* k if S is a degree k admissible surface. Then we obviously have the following

Gluing condition:

$$t_{(a_1,w,a_2)} = t_{(a_1',w',a_2')} \text{ for any paired turn types } (a_1, w, a_2) \text{ and } (a_1', w', a_2'). \tag{5.1}$$

Let $i(\alpha) \in I$ be the index such that the arc α lies on $\gamma_{i(\alpha)}$. Recall that r_i is the coefficient of γ_i in the chain c. Then we also have the following

Normalizing condition:

$$\sum_{a_2,w} t_{(a_1,w,a_2)} = r_{i(a_1)} \text{ for any } a_1 \text{ and } \sum_{a_1,w} t_{(a_1,w,a_2)} = r_{i(a_2)} \text{ for any } a_2. \tag{5.2}$$

Definition 5.1. We say an admissible surface S decomposed in the above way satisfying the gluing and normalizing conditions is in its *normal form.*

The discussion above shows that any (monotone) admissible surface for the chain c can be put in normal form after ignoring sphere components, applying homotopy and compression, during which $-\chi^-(S)$ does not increase.

Note that if S is already in normal form, then the sum of $\chi(\Sigma)$ over all components differs from $\chi(S)$ by the number of proper arcs in F, which is half of the total number of turns. Thus in general, for any S we have

$$\frac{-\chi^-(S)}{k} \geq \frac{1}{2} \sum_{(a_1,w,a_2)} t_{(a_1,w,a_2)} + \sum \frac{-\chi(\Sigma)}{k} = \frac{1}{2}\sum_i r_i|\gamma_i| + \sum \frac{-\chi(\Sigma)}{k} \tag{5.3}$$

where the last equality follows from the normalizing condition (5.2) and $|\gamma_i|$ denotes the number of arcs on the hyperbolic tight loop γ_i and is 0 for elliptic γ_i. Since the quantity $\frac{1}{2}\sum_i r_i|\gamma_i|$ is purely determined by the chain c, the key is to estimate $\sum \frac{-\chi(\Sigma)}{k}$.

This for example yields a simple estimate of scl_G of chains supported in vertex groups in terms of relative scl.

Lemma 5.2. *For any vertex v and any chain $c \in C_1(G_v)$, we have*

$$\mathrm{scl}_G(c) \geq \mathrm{scl}_{(G_v,\{G_e\}_{t(e)=v})}(c).$$

Proof. We assume c to be null-homologous in G, since otherwise the result is trivially true. By continuity, we may further assume c to be a rational chain after an arbitrarily small change. Represent c by elliptic tight loops supported in X_v. Suppose $f : S \to X$ is an admissible surface for c of degree k in normal form. Let S_v be the disjoint union of components supported in $N(X_v)$. Since there are no hyperbolic loops in our chain, each component of S_v has non-positive Euler characteristic, and equation (5.3) implies

$$\frac{-\chi^-(S)}{k} \geq \frac{-\chi(S_v)}{k} = \frac{-\chi^-(S_v)}{k}.$$

Note that S_v is admissible of degree k for c in G_v relative to the nearby edge groups. It follows that $-\chi^-(S_v)/k \geq 2 \cdot \mathrm{scl}_{(G_v,\{G_e\}_{t(e)=v})}(c)$. Combining this with the inequality above, the conclusion follows from Lemma 2.9 since S is arbitrary. □

5.2. **Lower bounds from linear programming duality.** We introduce a general strategy to obtain lower bounds of scl in graphs of groups relative to vertex groups using the idea of linear programming duality. This has been used by the first author in the special case of free products to obtain uniform sharp lower bounds of scl [Che18b] and in the case of Baumslag–Solitar groups to compute scl of certain families of chains [Che19].

Consider a rational chain $c = \sum r_i g_i$ with $r_i \in \mathbb{Q}_{>0}$ and $\underline{g} = \{g_i, i \in I\}$ consisting of finitely many *hyperbolic* elements represented by tight loops $\underline{\gamma} = \{\gamma_i, i \in I\}$. With notation as in the previous subsection, for each turn type (a_1, w, a_2), we assign a *non-negative cost* $c_{(a_1,w,a_2)} \geq 0$.

Suppose S is an admissible surface of degree k for c relative to vertex groups. Then S is by definition admissible (in the absolute sense) of degree k for some rational chain $c' = c + c_{ell}$ where

c_{ell} is a rational chain of elliptic elements, each of which can be assumed to have infinite order. Hence the normal form discussed in the previous subsection applies to S.

The assignment above induces a cost for each polygonal boundary by linearity, i.e. its cost is the sum of costs of its turns. This further induces a *non-negative* cost on each component Σ in the decomposition of S in normal form: the cost of Σ is the sum of costs of its polygonal boundaries.

Lemma 5.3. *Let S be any relative admissible surface for c of degree k in normal form. With the $t_{(a_1,w,a_2)}$ notation (normalized number of turns) in the estimate (5.3), if every disk component Σ in the normal form of S has cost at least 1, then the normalized total cost*

$$\sum_{(a_1,w,a_2)} t_{(a_1,w,a_2)}c_{(a_1,w,a_2)} \geq \frac{1}{2}\sum r_i|\gamma_i| - \frac{-\chi^-(S)}{k}.$$

Proof. By (5.3), it suffices to prove

$$\sum \frac{\chi(\Sigma)}{k} \leq \sum_{(a_1,w,a_2)} t_{(a_1,w,a_2)}c_{(a_1,w,a_2)}.$$

Note that, for each component Σ in the normal form of S, either $\chi(\Sigma) \leq 0$ which does not exceed its cost, or Σ is a disk component and $\chi(\Sigma) = 1$ which is also no more than its cost by our assumption. The desired estimate follows by summing up these inequalities and dividing by k. □

In light of Lemma 5.3, to get lower bounds of scl relative to vertex groups, the strategy is to come up with suitable cost assignments $c_{(a_1,w,a_2)}$ such that

(1) every possible disk component has cost at least 1; and
(2) one can use the gluing condition (5.1) and normalizing condition (5.2) to bound the quantity $\sum_{(a_1,w,a_2)} t_{(a_1,w,a_2)}c_{(a_1,w,a_2)}$ from above by a constant.

5.3. **Uniform lower bounds.** Now we use the duality method above to prove sharp uniform lower bounds of scl in graphs of groups relative to vertex groups. The results are subject to some local conditions introduced as follows.

Definition 5.4. Let H be a subgroup of G. For $2 \leq k < \infty$, an element $g \in G \setminus H$ is a *relative k-torsion* for the pair (G, H) if

$$gh_1 \dots gh_k = id \tag{5.4}$$

for some $h_i \in H$. For $3 \leq n \leq \infty$, we say the subgroup H is *n-relatively torsion-free* (*n-RTF*) in G if there is no relative k-torsion for all $2 \leq k < n$. Similarly, we say g is *n-RTF* in (G, H) if $g \in G \setminus H$ is not a relative k-torsion for any $2 \leq k < n$.

By definition, m-RTF implies n-RTF whenever $m \geq n$.

Example 5.5. If H is normal in G, then g is a relative k-torsion if and only if its image in G/H is a k-torsion. In this case, H is n-RTF if and only if G/H contains no k-torsion for all $k < n$, and in particular, H is ∞-RTF if and only if G/H is torsion-free. Concretely, the subgroup $H = 6\mathbb{Z}$ in $G = \mathbb{Z}$ is not n-RTF for all $n \geq 3$ since z^3 is a relative 2-torsion, where z is a generator of $\mathbb{Z}$; more generally, $H = m\mathbb{Z}$ is p_m-RTF if m is odd and p_m is the smallest prime factor of $m \in \mathbb{Z}_+$.

For a less trivial example, let S be an orientable closed surface of positive genus and let $g \in \pi_1(S)$ be an element represented by a simple closed curve. Then the cyclic subgroup $\langle g \rangle$ is ∞-RTF in $\pi_1(S)$. One can see this from either Lemma 5.15 or Lemma 5.18.

The equation (5.4) can be rewritten as

$$g \cdot \tilde{h}_1 g \tilde{h}_1^{-1} \cdot \ldots \cdot \tilde{h}_{k-1} g \tilde{h}_{k-1}^{-1} = \tilde{h}_k^{-1}, \tag{5.5}$$

where $\tilde{h}_i := h_1 \ldots h_i$. This is closely related to the notion of generalized k-torsion.

Definition 5.6. For $k \geq 2$, an element $g \neq id \in G$ is a *generalized k-torsion* if

$$g_1 g g_1^{-1} \ldots g_k g g_k^{-1} = id$$

for some $g_i \in G$.

If equation (5.4) holds with $\tilde{h}_k = h_1 h_2 \ldots h_k = id$, then g is a generalized k-torsion.

It is observed in [IMT19, Theorem 2.4] that a generalized k-torsion cannot have scl exceeding $1/2 - 1/k$ for a reason similar to Proposition 5.7 below. On the other hand, it is well known and easy to note that the existence of any generalized torsion is an obstruction for a group G to be bi-orderable, i.e. to admit a total order on G that is invariant under left and right multiplications.

The n-RTF condition is closely related to lower bounds of relative scl.

Proposition 5.7. *Let H be a subgroup of G. If*

$$\mathrm{scl}_{(G,H)}(g) \geq \frac{1}{2} - \frac{1}{2n},$$

then $g \in G$ is n-RTF in (G, H).

Proof. Suppose equation (5.5) holds for some $k \geq 2$. Then this gives rise to an admissible surface S in G for g of degree k relative to H, where S is a sphere with $k+1$ punctures: k of them each wraps around g once, and the other maps to $\tilde{h}_k$. This implies

$$\frac{1}{2} - \frac{1}{2n} \leq \mathrm{scl}_{(G,H)}(g) \leq \frac{-\chi(S)}{2k} = \frac{1}{2} - \frac{1}{2k},$$

and thus $k \geq n$. □

Conversely, the n-RTF condition implies a lower bound for relative scl in the case of graphs of groups.

Theorem 5.8. *Let $G = \mathcal{G}(\Gamma, \{G_v\}, \{G_e\})$ be a graph of groups. Let γ be a tight loop cut into arcs $a_1, \ldots, a_L$ by the edge spaces, where a_i is supported in a thickened vertex space $N(X_{v_i})$ and $v_1, e_1, \ldots, v_L, e_L$ form a loop in Γ with $o(e_i) = v_i$ and $t(e_i) = v_{i+1}$, indices taken mod L. Suppose for some $n \geq 3$, whenever $e_{i-1} = \bar{e}_i$, the winding number $w(a_i)$ of a_i is n-RTF in $(G_{v_i}, o_{e_i}(G_{e_i}))$. Then*

$$\mathrm{scl}_{(G,\{G_v\})}(g) \geq \frac{1}{2} - \frac{1}{n}.$$

Proof. Let S be a (monotone) relative admissible surface for γ of degree k. Put S in its normal form with components Σ. We follow the strategy and notation in Subsection 5.2 and assign costs in a way that does not depend on winding numbers. That is, for any $1 \leq i, j \leq L$, the cost $c_{(a_i,w,a_j)} = c_{ij}$ where

$$c_{ij} := \begin{cases} 1 - \frac{1}{n}, & \text{if } i < j \\ \frac{1}{n}, & \text{if } i \geq j. \end{cases}$$

Let t_{ij} be normalized the total number of turns of the form (a_i, w, a_j), i.e. $t_{ij} = \sum_w t_{(a_i,w,a_j)}$.

For any disk component Σ, let $a_{i_1}, \ldots a_{i_s}$ be the arcs of γ on the polygonal boundary of Σ in cyclic order; see the disk component in Figure 7. There are two cases:

(1) All $i_j = i$ for some i. Then we necessarily have $e_{i-1} = \bar{e}_i$, and $a_i \in G_{v_i} \backslash o_{e_i}(G_{e_i})$ since γ is tight. For each $1 \le j \le s$, let $w_j \in o_{e_i}(G_{e_i})$ be the winding number of the turn from a_{i_j} to $a_{i_{j+1}}$, where the j-index is taken mod s. Since Σ is a disk, we have $w(a_i)w_1 \cdots w(a_i)w_s = id \in G_{v_i}$. Then we must have $s \ge n$ since $w(a_i)$ is n-RTF in $(G_{v_i}, o_{e_i}(G_{e_i}))$ by assumption, and thus the cost

$$c(\Sigma) = s \cdot c_{ii} = \frac{s}{n} \ge 1.$$

(2) Some $i_j \neq i_{j'}$. Then there is some $j_m \neq j_M$ such that $i_{j_m} < i_{j_m+1}$ and $i_{j_M} > i_{j_M+1}$, where $j_m + 1$ and $j_M + 1$ are interpreted mod s. Hence the cost

$$c(\Sigma) \ge c_{j_m, j_m+1} + c_{j_M, j_M+1} = \left(1 - \frac{1}{n}\right) + \left(\frac{1}{n}\right) = 1.$$

In summary, we have $c(\Sigma) \ge 1$ for any disk component Σ. Hence by Lemma 5.3, we have

$$\frac{-\chi^-(S)}{2k} \ge \frac{L}{4} - \frac{1}{2}\sum_{ij} c_{ij} t_{ij}$$

since $|\gamma| = L$.

On the other hand, for any $1 \le i, j \le L$, we have $t_{ij} = t_{j-1,i+1}$ by the gluing condition (5.1) and $\sum_i t_{ij} = \sum_j t_{ji} = 1$ by the normalizing condition (5.2), indices taken mod L. We also have $t_{i,i+1} = 0$ since γ is tight and intersects edge spaces transversely. Thus

$$\sum_{i,j} c_{ij} t_{ij} = \sum_{i,j} \frac{1}{n} t_{ij} + \left(1 - \frac{2}{n}\right) \sum_{i<j} t_{ij} = \frac{L}{n} + \left(1 - \frac{2}{n}\right) \sum_{i<j} t_{ij}.$$

and

$$\begin{aligned}
2\sum_{i<j} t_{ij} &= \sum_{1 \le i < j \le L} t_{ij} + \sum_{1 \le i < j \le L} t_{j-1,i+1} \\
&= \sum_{\substack{1 \le i \le L-1 \\ 2 \le j \le L}} t_{ij} + \sum_{1 \le i \le L-1} t_{i,i+1} \\
&= \left[\sum_{1 \le i,j \le L} t_{ij} - \sum_i t_{i1} - \sum_j t_{Lj}\right] + [0] \\
&= L - 2,
\end{aligned}$$

where the first two equalities can be visualized in Figure 9.

Putting the equations above together, we have

$$\sum_{ij} c_{ij} t_{ij} = \frac{L}{n} + \left(1 - \frac{2}{n}\right) \frac{L-2}{2} = \frac{L}{2} - \left(1 - \frac{2}{n}\right),$$

and

$$\frac{-\chi^-(S)}{2k} \ge \frac{L}{4} - \frac{1}{2}\sum_{ij} c_{ij} t_{ij} = \frac{1}{2} - \frac{1}{n}$$

for any relative admissible surface S. Thus the conclusion follows from Lemma 2.9. □

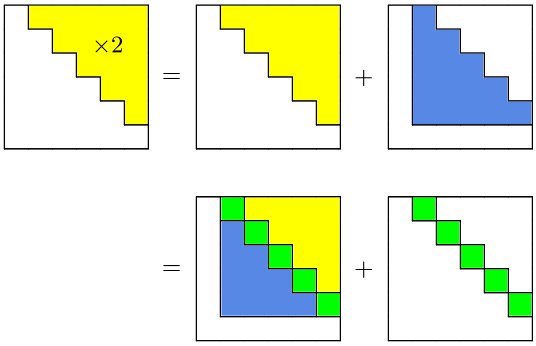

Figure 9. Visualization of the summation in the case $L = 6$, where the first equality uses the gluing condition $t_{ij} = t_{j-1,i+1}$

Theorem 5.9. *Let $G = \mathcal{G}(\Gamma, \{G_v\}, \{G_e\})$ be a graph of groups. Let $\{\Gamma_\lambda\}$ be a collection of mutually disjoint connected subgraphs of Γ and let G_λ be the graph of groups associated to Γ_λ. If for some $3 \le n \le \infty$ the inclusion of each edge group into vertex group is n-RTF, then*

$$\mathrm{scl}_G(g) \ge \mathrm{scl}_{(G,\{G_\lambda\})}(g) \ge \frac{1}{2} - \frac{1}{n}$$

unless $g \in G$ conjugates into some G_λ. Hence $(G, \{G_\lambda\})$ has a strong relative spectral gap $\frac{1}{2} - \frac{1}{n}$.

In particular,

$$\mathrm{scl}_G(g) \ge \mathrm{scl}_{(G,\{G_v\})}(g) \ge \frac{1}{2} - \frac{1}{n}$$

for any hyperbolic element $g \in G$.

Proof. Each hyperbolic element g is represented by a tight loop γ satisfying the assumptions of Theorem 5.8 since the inclusions of edge groups are n-RTF. This implies the special case where $\{\Gamma_\lambda\}$ is the set of vertices of Γ.

To see the general case, by possibly adding some subgraphs each consisting of a single vertex, we assume each vertex is contained in some Γ_λ. By collapsing each Γ_λ to a single vertex, we obtain a new splitting of G as a graph of groups where $\{G_\lambda\}$ is the new collection of vertex groups. Each new edge group is some G_e for some edge e of Γ connecting two Γ_λ's. Let Γ_λ be the subgraph containing the vertex $v = t(e)$, then (G_λ, G_v) is n-RTF by Lemma 5.18 in the next subsection and (G_v, G_e) is n-RTF by assumption. Thus (G_λ, G_e) is also n-RTF by Lemma 5.12. Therefore the edge group inclusions in this new splitting also satisfy the n-RTF condition, so the general case follows from the special case proved above. □

In the case of an amalgam $G = A \star_C B$, since left relatively convex implies ∞-RTF (Lemma 5.15), Theorem 5.9 implies [Heu19, Theorem 6.3], which is the main input to obtain gap 1/2 in all right-angled Artin groups in [Heu19]. We will discuss similar applications in Section 7.

For the moment, let us consider the case of Baumslag–Solitar groups.

Corollary 5.10. *Let $\mathrm{BS}(m,\ell) := \langle a, t \mid a^m = t a^\ell t^{-1} \rangle$ be the Baumslag–Solitar group, where $|m|, |\ell| \ge 2$. Let p_m and p_ℓ be the smallest prime factors of $|m|$ and $|\ell|$ respectively. Then $\mathrm{BS}(m,\ell)$ has strong spectral gap $1/2 - 1/\min(p_m, p_\ell)$ relative to the subgroup $\langle a \rangle$ if m, l are both odd. This estimate is sharp since for $g = a^{m/p_m} t a^{\ell/p_\ell} t^{-1} a^{-m/p_m} t a^{-\ell/p_\ell} t^{-1}$ we have*

$$\mathrm{scl}_{\mathrm{BS}(m,\ell)}(g) = \mathrm{scl}_{(\mathrm{BS}(m,\ell),\langle a \rangle)}(g) = 1/2 - 1/\min(p_m, p_\ell).$$

Proof. Let $n = \min(p_m, p_\ell)$. The Baumslag–Solitar group $\mathrm{BS}(m,\ell)$ is the HNN extension associated to the inclusions $\mathbb{Z} \xrightarrow{\times m} \mathbb{Z}$ and $\mathbb{Z} \xrightarrow{\times \ell} \mathbb{Z}$, which are both n-RTF. Thus the strong relative spectral gap follows from Theorem 5.9. The example achieving the lower bound follows from [Che19, Corollary 3.12], [Che18a, Proposition 5.6] and [Che19, Proposition 2.11]. □

If at least one of m and ℓ is even, the word g above has scl value 0, and thus one cannot have a strong relative spectral gap. However, we do have a (relative) spectral gap 1/12 by [CFL16, Theorem 7.8], which is sharp for example when $p_m = 2$ and $p_\ell = 3$. When $p_m = 2$ and $p_\ell \geq 3$, the smallest known positive scl in $\mathrm{BS}(m, \ell)$ is $1/4 - 1/2p_\ell$ achieved by $a^{m/p_m} t a^{\ell/p_\ell} t^{-1}$.

Question 5.11. If $p_m = 2$ and $p_\ell \geq 3$, does $\mathrm{BS}(m, \ell)$ have (relative) spectral gap $1/4 - 1/2p_\ell$?

5.4. **The n-RTF condition.** The goal of this subsection is to investigate the n-RTF condition that plays an important role in Theorem 5.8 and Theorem 5.9.

Let us start with some basic properties.

Lemma 5.12. *Suppose we have groups $K \leq H \leq G$.*

(1) If (G, K) is n-RTF, then so is (H, K);
(2) If $g \in G \setminus H$ is n-RTF in (G, H), then g is also n-RTF in (G, K);
(3) If both (G, H) and (H, K) are n-RTF, then so is (G, K).

Proof. (1) and (2) are clear from the definition. As for (3), suppose $gk_1 \ldots gk_i = id$ for some $1 \leq i \leq n$ where each $k_j \in K$. Then $g \in H$ since (G, H) is n-RTF, from which we get $g \in K$ since (H, K) is n-RTF. □

The n-RTF condition is closely related to orders on groups.

Definition 5.13. A subgroup H is *left relatively convex* in G if there is a total order on the left cosets G/H that is G-invariant, i.e. $gg_1H \prec gg_2H$ for all g if $g_1H \prec g_2H$.

The definition does not require G to be left-orderable, i.e. G may not have a total order invariant under the left G-action. Actually, if H is left-orderable, then H is left relatively convex in G if and only if G has a left G-invariant order $\prec$ such that H is convex, i.e. $h \prec g \prec h'$ for some $h, h' \in H$ implies $g \in H$. Many examples and properties of left relatively convex subgroups are discussed in [ADŠ18].

Example 5.14 ([ADŠ18])**.** Let G be a surface group, a pure braid group or a subgroup of some right-angled Artin group. Let H be any maximal cyclic subgroup of G, that is, there is no cyclic subgroup of G strictly containing H. Then H left relatively convex in G.

The n-RTF conditions share similar properties with the left relatively convex condition, and they are weaker.

Lemma 5.15. *If H is left relatively convex in G, then (G, H) is ∞-RTF.*

Proof. Suppose for some $g \in G$ we have $gh_1 \ldots gh_n = id$ for some $n \geq 2$ and $h_i \in H$ for all $1 \leq i \leq n$. Suppose $gH \succ H$. Then $gh_{n-1}gH \succ gh_{n-1}H = gH \succ H$ by left-invariance. By induction, we have $gh_1 \ldots gh_{n-1}gH \succ H$, but $gh_1 \ldots gh_{n-1}gH = gh_1 \ldots gh_nH = H$, contradicting our assumption. A similar argument shows that we cannot have $gH \prec H$. Thus we must have $g \in H$. □

The n-RTF condition has nice inheritance in graphs of groups (5.18). To prove it together with a more precise statement (Lemma 5.16), we first briefly introduce the reduced words of elements in graphs of groups. See [Ser80] for more details. For a graph of groups $G(\Gamma) = \mathcal{G}(\Gamma, \{G_v\}, \{G_e\})$, let $F(\Gamma)$ be the quotient group of $(\star G_v) \star F_E$ by relations $\bar{e} = e^{-1}$ and $et_e(g)e^{-1} = o_e(g)$ for any edge $e \in E$ and $g \in G_e$, where F_E is the free group generated by the edge set E. Let $P = (v_0, e_1, v_1, \ldots, e_k, v_k)$ be any oriented path (so $o(e_i) = v_{i-1}, t(e_i) = v_i$), and let $\mu = (g_0, \ldots, g_k)$ be a sequence of elements with $g_i \in G_{v_i}$. We say any word of the form $g_0 e_1 g_1 \cdots e_k g_k$ is *of type* (P, μ), and it is *reduced* if

(1) $k \geq 1$ and $g_i \notin \mathrm{Im}t_{e_i}$ whenever $\bar{e}_i = e_{i+1}$; or
(2) $k = 0$ and $g_0 \neq id$.

It is known that every reduced word represents a nontrivial element in $F(\Gamma)$. Fix any base vertex v_0, then $G(\Gamma)$ is isomorphic to the subgroup of $F(\Gamma)$ consisting of words of type (L, μ) for any oriented loop L based at v_0 and any μ. Moreover, any nontrivial element is represented by some reduced word of type (L, μ) as above.

Lemma 5.16. *With notation as above, let $g \in G = G(\Gamma)$ be an element represented by a reduced word of type (L, μ) with an oriented loop $L = (v_0, e_1, v_1, \ldots, e_j, v_j = v_0)$ and $\mu = (g_0, \ldots, g_j)$, $j \geq 1$. If j is odd, then g is ∞-RTF in (G, G_{v_0}); if j is even and $g_{j/2}$ is n-RTF in $(G_{v_{j/2}}, \mathrm{Im}t_{e_{j/2}})$ for some $n \geq 3$, then g is n-RTF in (G, G_{v_0}).*

Proof. If j is odd, then the projection $\bar{g}$ of g in the free group F_E is represented by a word of odd length, and thus must be of infinite order. It follows that the projection of $gh_1 \cdots gh_k$ is $\bar{g}^k$ for any $h_i \in G_{v_0}$ and $k > 0$, which must be nontrivial.

Now suppose $j = 2m$ is even and consider $w := gh_1 \cdots gh_k$ for some $1 \leq k < n$ and $h_i \in G_{v_0}$. We claim that there cannot be too much cancellation between the suffix and prefix of two nearby copies of g, more precisely, $g_m e_{m+1} \cdots e_j g_j h g_0 e_1 \cdots e_m g_m$ for any $h \in G_{v_0}$ can be represented by

(1) $g_m g'_m g_m$ for some $g'_m \in \mathrm{Im}t_{e_m}$; or
(2) a reduced word $g_m e_{m+1} \cdots g_{j-s-1} e_{j-s} g'_{j-s} e_{s+1} g_{s+1} \cdots e_m g_m$ with $0 \leq s < m$.

In fact, if either $e_j \neq \bar{e}_1$, or $e_j = \bar{e}_1$ and $g_j h g_0 \notin \mathrm{Im}t_{e_j}$, then we have case (2) with $s = 0$ and $g'_j = g_j h g_0$. If $e_j = \bar{e}_1$ and $g_j h g_0 \in \mathrm{Im}t_{e_j}$, then $v_{j-1} = v_1 = o(e_j)$ and we can replace $g_{e_{j-1}} e_j g_j h g_0 e_1 g_1$ by $g_{e_{j-1}} o_{e_j} t_{e_j}^{-1}(g_j h g_0) g_1 \in G_{v_{j-1}} = G_{v_1}$ to simplify w to a word of shorter length. This simplification procedure either stops in s steps with $s < m$ and we end up with case (2) or it continues until we arrive at $g_m o_{e_{m+1}}(g_m^*) g_m$ for some $g_m^* \in G_{e_{m+1}}$. Note that in the latter case, we must have $\bar{e}_m = e_{m+1}$ since the simplification continues all the way. Thus $g'_m := o_{e_{m+1}}(g_m^*) = t_{e_m}(g_m^*) \in \mathrm{Im}t_{e_m}$.

For each $1 \leq i \leq k$, write $w_i := e_{m+1} \cdots e_j g_j h_i g_0 e_1 \cdots e_m$ in a reduced form so that $g_m w_i g_m$ is of the form as in the claim above. Then a conjugate of w in $F(\Gamma)$ is represented by $g_m w_1 \cdots g_m w_k$. If $g_m w_i g_m$ is of the form (1) above for all i, then $w_i \in \mathrm{Im}t_{e_m}$ and $g_m w_1 \cdots g_m w_k \neq id$ since g_m is n-RTF in $(G_{v_m}, \mathrm{Im}t_{e_m})$ by assumption. Now suppose $i_1 < i_2 < \cdots < i_{k'}$ are the indices i such that $g_m w_i g_m$ is of the form (2) above, where $k' \geq 1$. Up to a cyclic conjugation, assume $i_{k'} = k$ and let $i_0 = 0$. We write

$$g_m w_1 \cdots g_m w_k = \tilde{g}_1 w_{i_1} \cdots \tilde{g}_k w_{i_{k'}},$$

where $\tilde{g}_s := g_m w_{i_{s-1}+1} \cdots g_m w_{i_s - 1} g_m$. Note by the definition of the i_j, each w_i that appears in $\tilde{g}_s$ (i.e. $i_{s-1} + 1 \leq i \leq i_s - 1$) lies in $\mathrm{Im}t_{e_m}$. It follows that each $\tilde{g}_s \in G_{v_m} \setminus \mathrm{Im}t_{e_m}$ since g_m is n-RTF in $(G_{v_m}, \mathrm{Im}t_{e_m})$ by assumption and $i_s - i_{s-1} - 1 < n$. Thus the expression above puts a conjugate of w in reduced form, and hence $w \neq id$. □

Corollary 5.17. *With notation as above, let $g \in G = G(\Gamma)$ be an element represented by a reduced word of type (L, μ) with an oriented loop $L = (v_0, e_1, v_1, \ldots, e_j, v_j = v_0)$ and $\mu = (g_0, \ldots, g_j)$, $j \geq 1$. Suppose for some $n \geq 3$ each g_i is n-RTF in (G_{v_i}, G_e) for any edge e adjacent to v_i. Then g is n-RTF in (G, G_{v_0}).*

Proof. This immediately follows from Lemma 5.16. □

Lemma 5.18. *Let $G(\Gamma) = \mathcal{G}(\Gamma, \{G_v\}, \{G_e\})$ be a graph of groups. If the inclusion of each edge group into an adjacent vertex group is n-RTF, then for any connected subgraph $\Lambda \subset \Gamma$, the inclusion of $G(\Lambda) := \mathcal{G}(\Lambda, \{G_v\}, \{G_e\}) \hookrightarrow G(\Gamma)$ is also n-RTF.*

Proof. The case where Λ is a single vertex v immediately follows from Corollary 5.17 by choosing v to be the base point in the definition of $G(\Gamma)$ as a subgroup of $F(\Gamma)$.

Now we prove the general case with the additional assumption that $\Gamma \setminus \Lambda$ contains only finitely many edges. We proceed by induction on the number of such edges. The assertion is trivially true for the base case $\Lambda = \Gamma$. For the inductive step, let e be some edge outside of Λ. If e is non-separating, then $G(\Gamma)$ splits as an HNN extension with vertex group $G(\Gamma - \{e\})$. In this case, the inclusion of the edge group G_e is n-RTF in $G_{o(e)}$, which is in turn n-RTF in $G(\Gamma - \{e\})$ by the single vertex case above. Thus by Lemma 5.12, the inclusion $G_e \hookrightarrow G(\Gamma - \{e\})$ is also n-RTF. The same holds for the inclusion of G_e into $G(\Gamma - \{e\})$ through $G_{t(e)}$. Therefore, using the single vertex case again for the HNN extension, we see that $(G(\Gamma), G(\Gamma - \{e\}))$ is n-RTF. Together with the induction hypothesis that $(G(\Gamma - \{e\}), G(\Lambda))$ is n-RTF, this implies that $(G(\Gamma), G(\Lambda))$ is n-RTF by Lemma 5.12. If e is separating, then $G(\Gamma)$ splits as an amalgam with vertex groups $G(\Gamma_1)$ and $G(\Gamma_2)$ such that $\Gamma = \Gamma_1 \sqcup \{e\} \sqcup \Gamma_2$ and $\Lambda \subset \Gamma_1$. The rest of the argument is similar to the previous case.

Finally the general case easily follows from what we have shown, as any $g \in G(\Gamma) \setminus G(\Lambda)$ can be viewed as an element in $G(\Gamma') \setminus G(\Lambda)$ for some connected subgraph Γ' of Γ with only finitely many edges in $\Gamma' \setminus \Lambda$. □

See Lemma 7.3 for a discussion on the n-RTF conditions in graph products. One can also use geometry to show that the peripheral subgroups of the fundamental group of certain compact 3-manifolds are 3-RTF; see Lemma 8.23.

5.5. **Quasimorphisms realizing the spectral gap for left relatively convex edge groups.** Recall from Definition 5.13 that a subgroup $H \leq G$ of a group H is called *left relatively convex*, if there is a left invariant order $\prec$ on the cosets $G/H = \{gH \mid g \in G\}$. This property has been studied in [ADŠ18]. Since left relatively convex subgroups are ∞-RTF by Lemma 5.15, Theorem 5.9 implies a sharp gap of $1/2$ for hyperbolic elements in graphs of groups where the edge groups are left relatively convex in the vertex groups. The aim of this subsection is to show the following result, which constructs explicit quasimorphisms realizing the gap $1/2$ and gives a completely different proof.

Theorem 5.19. *Let G be a graph of groups and let $g \in G$ be a hyperbolic element. If every edge group lies left relatively convex in its corresponding vertex groups then there is an explicit homogeneous quasimorphism $\phi\colon G \to \mathbb{R}$ such that $\phi(g) \geq 1$ and $D(\phi) \leq 1$.*

We will first prove this result for amalgamated free products and HNN extensions. The first case has been done in [Heu19, Theorem 6.3]. In both cases we will construct a letter-quasimorphism (see Definition 4.7) $\Phi\colon G \to \mathcal{A}$ and use Theorem 4.8.

We first recall the construction in the case of amalgamated free products. Let $G = A \star_C B$ be an amalgamated free product of A and B over the subgroup C and suppose that $C \leq A$ (resp. $C \leq B$) is left relatively convex with order $\prec_A$ (resp. $\prec_B$). Define a function sign_A on A by setting

$$\mathrm{sign}_A(a) = \begin{cases} -1 & \text{if } aC \prec_A C \\ 0 & \text{if } aC = C \\ 1 & \text{if } aC \succ_A C, \end{cases}$$

and define sign_B analogously. Every element $g \in G$ either lies in C or may be written as a unique product

$$g = a_0 b_0 \cdots a_n b_n$$

where possibly a_0 and b_n are the identity and all other $a_i \in A \setminus C$ and $b_i \in B \setminus C$.

Then we define a map $\Phi\colon G \to \mathcal{A}$ as follows: If $g \in C$ set $\Phi(g) = e$. If $g = a_0 b_0 \cdots a_n b_n$ set

$$\Phi(g) = \mathtt{a}^{\operatorname{sign}_A(a_0)} \mathtt{b}^{\operatorname{sign}_B(b_0)} \cdots \mathtt{a}^{\operatorname{sign}_A(a_n)} \mathtt{b}^{\operatorname{sign}_B(b_n)}.$$

Lemma 5.20 ([Heu19, Theorem 6.3 and Lemma 6.1])**.** *Let $G = A \star_C B$ be an amalgamated free product where C lies left relatively convex in both A and B. Then the map $\Phi\colon G \to F(\{\mathtt{a}, \mathtt{b}\})$ defined as above is a letter-quasimorphism. If $g \in G$ is a hyperbolic element then there are elements $\mathtt{b}_l, \mathtt{b}, \mathtt{b}_r$ such that $\Phi(g^n) = \mathtt{b}_l \mathtt{b}^n \mathtt{b}_r$.*

We now describe a similar construction for HNN extensions. Let V be a group and let $\phi\colon A \to B$ be an isomorphism between two subgroups $A, B \leq V$ of V. Let

$$G = V\star_\phi := \langle V, r \mid \phi(a) = tat^{-1}, a \in A\rangle$$

be the associated HNN extension with stable letter t.

Our construction makes use of a function $\operatorname{sign}_A\colon G \to \{-1, 0, 1\}$ as follows. If A, B are left relatively convex in V, then a result of Antolín–Dicks–Šunić [ADŠ18, Theorem 14] shows that A is also left relatively convex in G. Define $\operatorname{sign}_A\colon G \to \{-1, 0, 1\}$ as

$$\operatorname{sign}_A(g) = \begin{cases} -1 & \text{if } gA \prec A \\ 0 & \text{if } gA = A \\ 1 & \text{if } gA \succ A. \end{cases}$$

Observe that for every $g \in G$ and $a \in A$, $\operatorname{sign}_A(g) = \operatorname{sign}_A(ga) = \operatorname{sign}_A(ag)$, i.e. sign_A is invariant under both left and right multiplications by A.

Britton's lemma asserts that every element $g \in G$ in the HNN extension may be written as

$$g = v_0 t^{\epsilon_0} \cdots t^{\epsilon_{n-1}} v_n$$

where $\epsilon_i \in \{+1, -1\}$, $v_i \in V$ and there is no subword tat^{-1} with $a \in A$ or $t^{-1}bt$ with $b \in B$.

Such an expression is unique in the following sense. If $v'_0 t^{\epsilon'_0} \cdots t^{\epsilon'_{n'-1}} v'_{n'}$ is another such expression then $n = n'$, $\epsilon_i = \epsilon'_i$ for all $i \in \{0, \cdots, n-1\}$, and

$$v_i = h_i^l v'_i h_i^r,$$

where

$$\begin{cases} h_i^l = e & \text{if } i = 0, \\ h_i^l \in A & \text{if } \epsilon_{i-1} = 1, i > 0 \text{ and} \\ h_i^l \in B & \text{if } \epsilon_{i-1} = -1, i > 0 \end{cases}$$

and

$$\begin{cases} h_i^r = e & \text{if } i = n, \\ h_i^r \in A & \text{if } \epsilon_i = -1, i < n \text{ and} \\ h_i^r \in B & \text{if } \epsilon_i = 1, i < n. \end{cases}$$

We define a letter-quasimorphism $\Phi\colon G \to F(\{\mathtt{a}, \mathtt{b}\})$ as follows. If $g \in V$, set $\Phi(g) = e$. Otherwise, express $g = v_0 t^{\epsilon_0} \cdots t^{\epsilon_{n-1}} v_n$ in the above form. Then set

$$\Phi(g) = \mathtt{a}\mathtt{b}^{\epsilon_0} \mathtt{a}^{\operatorname{sign}_A(\tilde{v}_1)} \mathtt{b}^{\epsilon_1} \cdots \mathtt{a}^{\operatorname{sign}_A(\tilde{v}_{n-1})} \mathtt{b}^{\epsilon_{n-1}} \mathtt{a}^{-1}$$

where $\tilde{v}_i = t_i^l v_i t_i^r$ with

$$t_i^l = \begin{cases} t^{-1} & \text{if } \epsilon_{i-1} = -1 \\ e & \text{else,} \end{cases}$$

and

$$t_i^r = \begin{cases} t & \text{if } \epsilon_i = +1 \\ e & \text{else.} \end{cases}$$

The map Φ is well defined. If $v_0' t^{\epsilon_0'} \cdots t^{\epsilon_{n-1}'} v_n'$ is another such expression, then we know that $\epsilon_i' = \epsilon_i$, and thus the b-terms agree.

We are left to show that the sign of $t_i^l v_i t_i^r$ and $t_i^l v_i' t_i^r$ agrees, where we know that $v_i = h_i^l v_i' h_i^r$ for some h_i^l, h_i^r as above.

If $\epsilon_{i-1} = 1$, then

$$\mathrm{sign}_A(t_i^l v_i t_i^r) = \mathrm{sign}_A(h_i^l v_i' h_i^r t_i^r) = \mathrm{sign}_A(t_i^l v_i' h_i^r t_i^r)$$

using that t_i^l is trivial, $h_i^l \in A$ and that sign_A is invariant under left multiplication. If $\epsilon_{i-1} = -1$, then

$$\mathrm{sign}_A(t_i^l v_i t_i^r) = \mathrm{sign}_A(t^{-1} h_i^l v_i' h_i^r t_i^r) = \mathrm{sign}_A\big(\big(t^{-1} h_i^l t\big)\, t^{-1} v_i' h_i^r t_i^r\big) = \mathrm{sign}_A(t_i^l v_i' h_i^r t_i^r)$$

using that $t_i^l = t^{-1}$, $h_i^l \in B$, $t^{-1} h_i^l t \in A$ and that sign_A is invariant under left multiplication. In each case we see that

$$\mathrm{sign}_A(t_i^l v_i t_i^r) = \mathrm{sign}_A(t_i^l v_i' h_i^r t_i^r).$$

By an analogous argument for the right hand side of v_i and v_i' we see that

$$\mathrm{sign}_A(t_i^l v_i t_i^r) = \mathrm{sign}_A(t_i^l v_i' t_i^r).$$

Thus Φ is indeed well defined.

Lemma 5.21. *Let V be a group with left relatively convex subgroups $A, B \leq V$. Let $\phi\colon A \to B$ be an isomorphism between A and B and let $G = V\star_\phi$ be the associated HNN extension. Then the map $\Phi\colon G \to \mathcal{A}$ defined as above is a letter-quasimorphism. If g is an hyperbolic element, then there are $b_l, b_r, b \in \mathcal{A}$ such that $\Phi(g^m) = b_l b^{m-1} b_r$ for all $m \in \mathbb{N}$ where $b \in \mathcal{A}$ is nontrivial and has even length.*

We will show that for every $g_1, g_2, g_3 \in G$ with $g_1 \cdot g_2 \cdot g_3 = 1_G$ either (1) or (2) of Definition 4.7 holds by realizing Φ as a map defined on the Bass–Serre tree associated to the HNN extension, see also the proof of [Heu19, Lemma 6.1].

Let T be a tree with vertex set $V(\mathrm{T}) = \{gV \mid g \in G\}$ and edge set

$$E(\mathrm{T}) = \{(gV, gtV) \mid g \in G\} \cup \{(gV, gt^{-1}V) \mid g \in G\}.$$

Define $o, \tau\colon E(\mathrm{T}) \to V(\mathrm{T})$ by setting $o((gV, gtV)) = gV$, $\tau((gV, gtV)) = gtV$, $o((gV, gt^{-1}V)) = gV$, $\tau((gV, gt^{-1}V)) = gt^{-1}V$. Moreover, set $\overline{(gV, gtV)} = (gtV, gV)$ and $\overline{(gV, gt^{-1}V)} = (gt^{-1}V, gV)$. It is well known that T is a tree and that G acts on T with vertex stabilizers conjugate to V and edge stabilizers conjugate to A. For what follows, we will define two maps $\mathrm{sign}_t\colon E(\mathrm{T}) \to \{1, -1\}$ and $\mu\colon E(\mathrm{T}) \to G/A$ on the set of edges. We set

- $\mathrm{sign}_t(e) = 1$ if $e = (gV, gtV)$ and $\mathrm{sign}_t(e) = -1$ if $e = (gV, gt^{-1}V)$. In the first case we call e *positive* and in the second case *negative.*
- $\mu(e) = gtA$ if $e = (gV, gtV)$, and $\mu(e) = gA$ if $e = (gV, gt^{-1}V)$.

Claim 5.22. *Both sign_t and μ are well defined.*

Proof. For sign_t there is nothing to prove. For μ, observe that if $(gV, gtV) = (g'V, g'tV)$, then there is an element $b \in B$ such that $g = g'b$. Thus $gtA = g'btA = g'tA$, using that $tAt^{-1} = B$ in G. Similarly, we see that μ is also well defined for negative edges. ☐

Claim 5.23. *Let $e \in E(\mathrm{T})$ be an edge. Then $\mathrm{sign}_t(\bar{e}) = -\mathrm{sign}_t(e)$ and $\mu(e) = \mu(\bar{e})$.*

Proof. Suppose that $e = (gV, gtV)$. Hence $\mathrm{sign}_t(e) = 1$ and $\mu(e) = gtA$. We see that $\bar{e} = (gtV, gV) = ((gt)V, (gt)t^{-1}V)$. Thus $\mathrm{sign}_t(\bar{e}) = -1$ and $\mu(\bar{e}) = gtA$. The case where e is an edge of the form $(gV, gt^{-1}V)$ is analogous. ☐

A *reduced path in* T is a sequence $\wp = (e_1, \dots e_n)$, $e_i \in E(\mathrm{T})$ of edges such that $\tau(e_i) = o(e_{i+1})$ for every $i \in \{1, \dots, n-1\}$, without backtracking. For what follows, $\mathcal{P}$ will be the set of all reduced paths. We also allow the empty path.

We define the following map $\Xi \colon \mathcal{P} \to \mathcal{A}$ assigning an alternating word to each path of edges.

If $\wp$ is empty, set $\Xi(\wp) = e$. Otherwise, suppose that $\wp = (e_1, \dots, e_n)$. Then define $\Xi(\wp) \in F(\{\mathtt{a}, \mathtt{b}\})$ as

$$\mathtt{a}\mathtt{b}^{\mathrm{sign}_t(e_1)}\mathtt{a}^{\mathrm{sign}_A(\mu(e_1)^{-1}\mu(e_2))}\mathtt{b}^{\mathrm{sign}_t(e_2)} \cdots \mathtt{b}^{\mathrm{sign}_t(e_{n-1})}\mathtt{a}^{\mathrm{sign}_A(\mu(e_{n-1})^{-1}\mu(e_n))}\mathtt{b}^{\mathrm{sign}_t(e_n)}\mathtt{a}^{-1}.$$

Claim 5.24. *$\Xi \colon \mathcal{P} \to \mathcal{A}$ has the following properties:*

(i) For any $\wp \in \mathcal{P}$ and $g \in G$ we have $\Xi({}^g\wp) = \Xi(\wp)$, where ${}^g\wp$ is the translate of $\wp$ by $g \in G$.

(ii) Let $\wp_1, \wp_2$ be two reduced paths such that the last edge in $\wp_1$ is e_1, the first edge in $\wp_2$ is e_2 such that $\tau(e_1) = o(e_2)$ and such that $e_1 \neq \bar{e}_2$. Then

$$\Xi(\wp_1 \cdot \wp_2) = \Xi(\wp_1)\mathtt{a}^{sign_A(\mu(e_1)^{-1}\mu(e_2))}\Xi(\wp_2)$$

as reduced words, where $\wp_1 \cdot \wp_2$ denotes the concatenation of paths.

(iii) For any $g \in G$ let $\wp(g)$ be the unique path in T *connecting V with gV. Then $\Phi(g) = \Xi(\wp(g))$, where Φ is the map of interest in Lemma 5.21.*

Proof. To see (i), denote by ${}^g e$ the translate of an edge e by the element $g \in G$. Then observe that $\mu({}^g e) = g\mu(e)$ and that $\mathrm{sign}_t({}^g e) = \mathrm{sign}_t(e)$. Thus for every sequence $({}^g e_1, {}^g e_2)$ of edges we see that $\mu({}^g e_1)^{-1}\mu({}^g e_2) = \mu(e_1)^{-1}\mu(e_2)$. Property (ii) follows immediately from the definition. For (iii), let $g = v_0 t^{\epsilon_0} \cdots t^{\epsilon_{n-1}} v_n$ as above. Then the unique path between V and gV may be described as

$$((v_0 V, v_0 t^{\epsilon_0} V), (v_0 t^{\epsilon_0} v_1 V, v_0 t^{\epsilon_0} v_1 t^{\epsilon_1} V), \cdots, (v_0 t^{\epsilon_0} v_1 \cdots t^{\epsilon_{n-2}} v_{n-1} V, gV)).$$

We conclude by multiple applications of (ii). □

We can now prove Lemma 5.21.

Proof of Lemma 5.21. Since $\mathrm{sign}_A(g^{-1}) = -\mathrm{sign}_A(g)$, it is easy to see that Φ is alternating, i.e. $\Phi(g^{-1}) = \Phi(g)^{-1}$. Let $g \in G$ be a hyperbolic element. Up to conjugation, g may be written as a cyclically reduced word $g = v_0 t^{\epsilon_0} \cdots v_n t^{\epsilon_n}$. Thus $t^{\epsilon_n} v_0 t^{\epsilon_0}$ is reduced. We observe that

$$\Phi(g^m) = \mathtt{a}\left(\mathtt{b}^{\epsilon_0}\mathtt{a}^{\mathrm{sign}_A(\tilde{v}_1)} \cdots \mathtt{a}^{\mathrm{sign}_A(\tilde{v}_n)}\mathtt{b}^{\epsilon_n}\mathtt{a}^{\mathrm{sign}_A(\tilde{v}_0)}\right)^{m-1}\mathtt{b}^{\epsilon_n}\mathtt{a}^{\mathrm{sign}_A(\tilde{v}_1)} \cdots \mathtt{a}^{\mathrm{sign}_A(\tilde{v}_n)}\mathtt{b}^{\epsilon_n}\mathtt{a}^{-1}$$

where $\tilde{v_i}$ is defined as before. This produces desired $b_l, b_r, b \in \mathcal{A}$ such that $\Phi(g^m) = b_l b^{m-1} b_r$ for all $m \in \mathbb{N}$.

It remains to show that Φ is a letter-quasimorphism. Let $g, h \in G$. First, suppose that V, gV and ghV lie on a geodesic segment on T. If gV lies in the middle of this segment, then there are paths $\wp_1$ and $\wp_2$ such that $\wp(g) = \wp_1$, ${}^g\wp(h) = \wp_2$ and $\wp(gh) = \wp_1 \cdot \wp_2$. Let e_1 be the last edge of $\wp_1$ and e_2 be the first edge of $\wp_2$. Using Claim 5.24 points (i) and (ii) we see that Φ behaves as a letter-quasimorphism on such g, h.

Now suppose that V, gV and ghV do not lie on a common geodesic segment. Then there are nontrivial paths $\wp_1, \wp_2, \wp_3 \in \mathcal{P}$ with initial edges e_1, e_2, e_3 satisfying $o(e_1) = o(e_2) = o(e_3)$ and $e_i \neq e_j$ for $i \neq j$ such that

$$\wp(g) = \wp_1^{-1} \cdot \wp_2 \text{ , } {}^g\wp(h) = \wp_2^{-1} \cdot \wp_3 \text{ , and } {}^{gh}\wp((gh)^{-1}) = \wp_3^{-1} \cdot \wp_1.$$

By Claim 5.24 we obtain that

$$\begin{aligned}
\Phi(g) &= c_1^{-1}\mathtt{a}^{\mathrm{sign}_A(\mu(e_1)^{-1}\mu(e_2))}c_2\\
\Phi(h) &= c_2^{-1}\mathtt{a}^{\mathrm{sign}_A(\mu(e_2)^{-1}\mu(e_3))}c_3\\
\Phi(gh)^{-1} &= c_3^{-1}\mathtt{a}^{\mathrm{sign}_A(\mu(e_3)^{-1}\mu(e_1))}c_1
\end{aligned}$$

Not all of the above signs can be the same. Indeed suppose that

$$\mathrm{sign}_A(\mu(e_1)^{-1}\mu(e_2)) = \mathrm{sign}_A(\mu(e_2)^{-1}\mu(e_3)) = \mathrm{sign}_A(\mu(e_3)^{-1}\mu(e_1)) = 1.$$

Then $\mu(e_1)^{-1}\mu(e_2)A \succ A$ and $\mu(e_2)^{-1}\mu(e_3)A \succ A$, thus by transitivity and invariance of the order

$$\mu(e_1)^{-1}\mu(e_3)A = \mu(e_1)^{-1}\mu(e_2)\mu(e_2)^{-1}\mu(e_3)A \succ \mu(e_1)^{-1}\mu(e_2)A \succ A.$$

But then $\mu(e_3)^{-1}\mu(e_1)A \prec A$, contradicting $\mathrm{sign}_A(\mu(e_3)^{-1}\mu(e_1)) = 1$. Similarly the signs cannot all be -1. Thus Φ is a letter-quasimorphism. □

We may now prove Theorem 5.19.

Proof. Let G be the fundamental group of a graph of groups such that the edge groups are left relatively convex. and let $g \in G$ be a hyperbolic element. G arises as a succession of amalgamated free products and HNN extensions. In particular, similarly to the proof of Lemma 5.18, we may write G either as an amalgamated free product or an HNN extension such that g is hyperbolic. By a result of Antolín–Dicks–Šunić [ADŠ18, Theorem 14], the edge groups of this HNN extension or amalgamated free product are left relatively convex. Lemmas 5.20 and 5.21 assert that there is a letter-quasimorphism $\Phi\colon G \to F(\{\mathtt{a},\mathtt{b}\})$ such that $\Phi(g^n) = b_l b^{n-1} b_r$ for all $n \geq 1$ and where $b \in F(\{\mathtt{a},\mathtt{b}\})$ is not a proper power of $\mathtt{a}$ or $\mathtt{b}$. We conclude using Theorem 4.8. □

6. Scl of vertex and edge groups elements

To promote relative spectral gap of graphs of groups to spectral gap in the absolute sense, one needs to further control scl in vertex groups. For a graph of groups $G = \mathcal{G}(\Gamma, \{G_v\}, \{G_e\})$, the goal of this section is to characterize scl_G of chains in vertex groups in terms of $\{\mathrm{scl}_{G_v}\}_v$.

Let us start with simple examples. For a free product $G = A \star B$, we know that G has a retract to each factor, and thus $\mathrm{scl}_G(a) = \mathrm{scl}_A(a)$ for any $a \in A$. This is no longer true in general for amalgams.

Example 6.1. Let S be a closed surface of genus $g > 4$. Let γ be a separating simple closed curve that cuts S into S_A and S_B, where S_A has genus $g-1$ and S_B has genus 1. Let a be an element represented by a simple closed loop α in S_A that bounds a twice-punctured torus S_m with γ; see Figure 10. Then $G = \pi_1(S)$ splits as $G = A \star_C B$, where $A = \pi_1(S_A)$, $B = \pi_1(S_B)$ and C is the cyclic group generated by an element represented by γ. In this case, the element a is supported in A and the corresponding loop α does bound a surface S_ℓ in S_A of genus $g-2$, which is actually a retract of S_A and we have $\mathrm{scl}_A(a) = g - 5/2$. However, α also bounds a genus two surface from the B side, which is the union of S_B and S_m, showing that $\mathrm{scl}_G(a) \leq 3/2$, which is smaller than $\mathrm{scl}_A(a)$ since $g > 4$.

The example above shows that one can use chains in edge groups to adjust the given chain in vertex groups to a better one before evaluating it in individual vertex groups. Actually, scl is obtained by making the best adjustment of this kind.

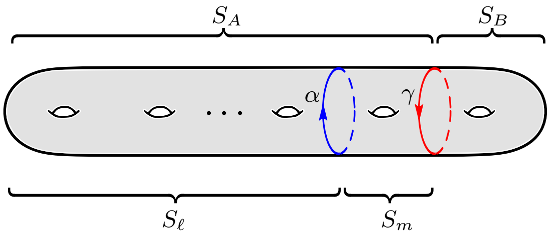

FIGURE 10. Illustration of Example 6.1

Theorem 6.2. *Let $G = \mathcal{G}(\Gamma, \{G_v\}, \{G_e\})$ be a graph of groups with $\Gamma = (V, E)$. For any finite collection of chains $c_v \in C_1^H(G_v)$, we have*

$$\mathrm{scl}_G(\sum_v c_v) = \inf \sum_v \mathrm{scl}_{G_v}(c_v + \sum_{t(e)=v} c_e), \tag{6.1}$$

where the infimum is taken over all finite collections of chains $c_e \in C_1^H(G_e)$ satisfying $c_e + c_{\bar{e}} = 0$ for each $e \in E$.

Proof. By a simple homology calculation using the Mayer–Vietoris sequence, we observe that $\|\mathrm{scl}_G(\sum_v c_v)\|$ is finite if and only if the infimum is. Thus we will assume both to be finite in the sequel. We will prove the equality for an arbitrary collection of rational chains c_v. Then the general case will follow by continuity.

Let X be the standard realization of G as in Subsection 2.3. Represent each chain c_v by a rational formal sum of elliptic tight loops in the corresponding vertex space X_v. Let $f : S \to X$ be any admissible surface in normal form (see Definition 5.1) of degree n for the rational chain $\sum c_v$ in $C_1^H(G)$. For each $v \in V$, let S_v be the union of components in the decomposition of S that are supported in the thickened vertex space $N(X_v)$. Note that S_v only has loop boundary since our chain is represented by elliptic loops. Moreover, any loop boundary supported in some edge space is obtained from cutting S along edge spaces. For each edge e with $t(e) = v$, let $c_e \in C_1^H(G_e)$ be the integral chain that represents the union of loop boundary components of S_v supported in X_e. Then S_v is admissible of degree n for the chain $c_v + \frac{1}{n}\sum_{t(e)=v} c_e$ in $C_1^H(G_v)$ for all $v \in V$. See Figure 11. Note that we must have $c_e + c_{\bar{e}} = 0$ since loops in c_e and $c_{\bar{e}}$ are paired and have opposite orientations. Since S is in normal form and $\sqcup S_v$ has no polygonal boundary, there are no disk components and hence we have

$$\frac{-\chi(S_v)}{2n} = \frac{-\chi^-(S_v)}{2n} \geq \mathrm{scl}_{G_v}(c_v + \sum_{t(e)=v} c_e), \quad \text{and}$$

$$\frac{-\chi(S)}{2n} = \sum_v \frac{-\chi(S_v)}{2n} \geq \sum_v \mathrm{scl}_{G_v}(c_v + \sum_{t(e)=v} c_e).$$

Since S is arbitrary, this proves the "$\geq$" direction in (6.1).

Conversely, consider any collection of chains $c_e \in C_1^H(G_e)$ satisfying $c_e + c_{\bar{e}} = 0$. With an arbitrarily small change of $\sum_v \mathrm{scl}_{G_v}(c_v + \sum_{t(e)=v} c_e)$, we replace this collection by another with the additional property that each c_e is a rational chain. This can be done since each c_v is rational. For each $v \in V$, let S_v be any admissible surface for the rational chain $c_v + \sum_{e:\ t(e)=v} c_e$. By taking suitable finite covers, we may assume all S_v to be of the same degree n. Since $c_e + c_{\bar{e}} = 0$ for all $e \in E$, the union $\sqcup_v S_v$ is an admissible surface of degree n for a chain equivalent to $\sum_v c_v$ in $C_1^H(G)$. Since the S_v and the collection c_e are arbitrary, this proves the other direction of (6.1). $\square$

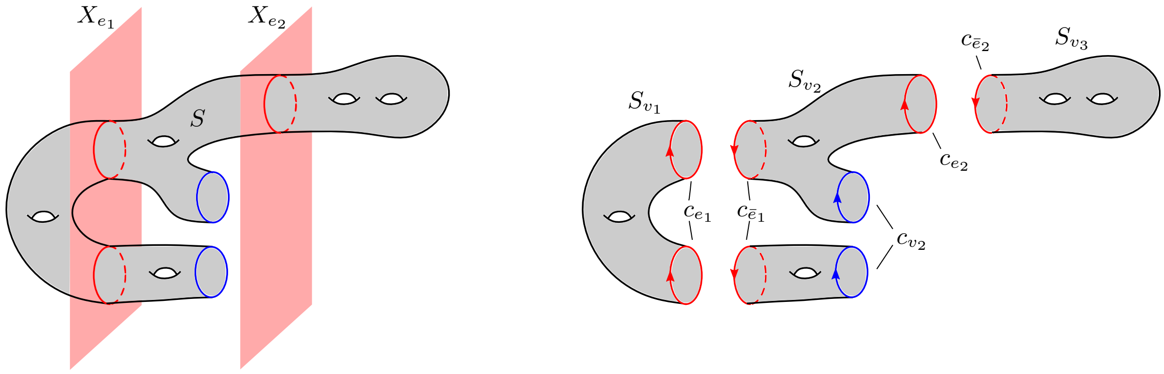

FIGURE 11. This is an example where $\sum c_v = c_{v_2}$ is supported in a single vertex group G_{v_2}, represented as the formal sum of the blue loops in X_{v_2}. On the left we have an admissible surface S for c_{v_2} of degree 1. The edge spaces X_{e_1}, X_{e_2} cut S in to $S_{v_1}, S_{v_2}, S_{v_3}$ shown on the right. The edges are oriented so that $t(e_1) = v_1$ and $t(e_2) = v_2$. Then S_{v_1} is admissible for its boundary, which is the chain c_{e_1}, and similarly S_{v_3} is admissible for $c_{\bar{e}_2}$,. The surface S_{v_2} is admissible for $c_{v_2}+c_{\bar{e}_1}+c_{e_2}$, where $c_{\bar{e}_1} = -c_{e_1}$ due to opposite orientations induced from S_{v_2} and S_{v_1}, and similarly $c_{e_2} = -c_{\bar{e}_2}$. Thus the sum of all boundary components of $\sqcup_i S_{v_i}$ is equal to c_{v_2} in $C_1^H(G)$.

Remark 6.3. If some $e \in E$ has $\mathrm{scl}_{G_e} \equiv 0$ (e.g. when G_e is amenable), then $\mathrm{scl}_{t(e)}$ and scl_G both vanish on $B_1^H(G_e)$ by monotonicity. Given this, the typically infinite-dimensional space $C_1^H(G_e)$ in Theorem 6.2 can be replaced by the quotient $C_1^H(G_e)/B_1^H(G_e) \cong H_1(G_e, \mathbb{R})$, which is often (e.g. when G_e is finitely generated) finite-dimensional, for which Theorem 6.2 is still valid. Thus if all edge groups have vanishing scl and scl_{G_v} is understood in each finite-dimensional subspace of $C_1^H(G_v)$ for all v, then scl_G in vertex groups can be practically understood by equation (6.1), which is a convex programming problem.

Corollary 6.4. *Let $G = A \star_C B$ be an amalgam. If C has vanishing scl and $H_1(C; \mathbb{R}) = 0$, then $\mathrm{scl}_G(a) = \mathrm{scl}_A(a)$ for any $a \in C_1^H(A)$.*

Proof. For any chain $c \in C_1^H(C)$, we have $\mathrm{scl}_C(c) = 0$ by our assumption, and thus $\mathrm{scl}_A(c) = \mathrm{scl}_B(c) = 0$ by monotonicity of scl. The conclusion follows readily from Theorem 6.2. ☐

It is clear from Example 6.1 that the assumption $H_1(C; \mathbb{R}) = 0$ is essential in Corollary 6.4.

By inclusion of edge groups into vertex groups, apparently Theorem 6.2 also applies to chains supported in edge groups. For later applications, we would like to carry out the characterization of scl in edge groups carefully, where we view scl as a *degenerate norm*.

Definition 6.5. A *degenerate norm* $\|\cdot\|$ on a vector space V is a pseudo-norm on a linear subspace V^f, called the *domain* of $\|\cdot\|$, and is $+\infty$ outside V^f. The unit norm ball B of $\|\cdot\|$ is the (convex) set of vectors v with $\|v\| \le 1$. The *vanishing locus* V^z is the subspace consisting of vectors v with $\|v\| = 0$. Note that $V^z \subset B \subset V^f$.

In the sequel, norms refer to degenerate norms unless emphasized as genuine norms. A norm in a finite-dimensional space with rational vanishing locus automatically has a "spectral gap".

Lemma 6.6. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. If the vanishing locus V^z is a rational subspace, then $\|\cdot\|$ satisfies a gap property on $\mathbb{Z}^n$: there exists $C > 0$ such that either $\|P\| = 0$ or $\|P\| \ge C$ for all $P \in \mathbb{Z}^n$.*

Proof. Extend a rational basis $e_1, \dots, e_m$ of V^z to a rational basis $e_1, \dots, e_n$ of $\mathbb{R}^n$, where $m \le n$. Then there is some integer $N > 0$ such that any $P = \sum_i P_i e_i \in \mathbb{Z}^n$ has $NP_i \in \mathbb{Z}$ for all i. The restriction of $\|\cdot\|$ to the subspace spanned by $e_{m+1}, \dots, e_n$ has trivial vanishing locus and thus there is a constant $C > 0$ such that $\|\sum_{j=m+1}^n Q_j e_j\| \ge NC$ for any integers $Q_{m+1}, \dots, Q_n$ unless they all vanish. Therefore, if $P = \sum_{i=1}^n P_i e_i \in \mathbb{Z}^n \setminus V^z$, then $\|P\| = \|\sum_{j=m+1}^n NP_j e_j\|/N \ge C$. □

Recall from Subsection 2.1 that scl_G is a (degenerate) norm on $C_1^H(G)$. If G is a subgroup of $\tilde{G}$, then $\mathrm{scl}_{\tilde{G}}$ also restricts to a norm on $C_1^H(G)$.

Let us set up some notation. For a graph of groups $G = \mathcal{G}(\Gamma, \{G_v\}, \{G_e\})$ with $\Gamma = (V, E)$. For each vertex $v \in V$, let $C_v := \bigoplus_{t(e)=v} C_1^H(G_e)$ be the space parameterizing chains in edge groups adjacent to v. Let $C_E := \bigoplus_{\{e,\bar{e}\}} C_1^H(G_e)$ parameterize chains in all edge groups.

Define $\|(c_{(e,\bar{e})})\|_E := \mathrm{scl}_G(\sum c_{e,\bar{e}})$ for any $(c_{(e,\bar{e})}) \in C_E$. Equivalently, $\|\cdot\|_E$ is the pullback of scl_G via

$$\bigoplus_{\{e,\bar{e}\}} C_1^H(G_e) \longrightarrow \bigoplus_{\{e,\bar{e}\}} C_1^H(G) \xrightarrow{+} C_1^H(G),$$

where the former map is inclusion on each summand and the latter map takes the summation.

Similarly, for each vertex $v \in V$, we pull back scl_{G_v} to get a norm $\|\cdot\|_v$ on $C_v = \bigoplus_{t(e)=v} C_1^H(G_e)$ via the composition

$$\bigoplus_{e:\ t(e)=v} C_1^H(G_e) \xrightarrow{\oplus t_{e*}} \bigoplus_{e:\ t(e)=v} C_1^H(G_v) \xrightarrow{+} C_1^H(G_v),$$

Note that $\bigoplus_{v\in V} C_v$ is naturally isomorphic to $\bigoplus_{\{e,\bar{e}\}} (C_1^H(G_e) \oplus C_1^H(G_{\bar{e}}))$ which has a surjective map π to C_E whose restriction on any summand is

$$C_1^H(G_e) \oplus C_1^H(G_{\bar{e}}) = C_1^H(G_e) \oplus C_1^H(G_e) \xrightarrow{+} C_1^H(G_e).$$

The kernel of π is exactly the collections of chains c_e over which we take infimum in equation (6.1).

Equipping $\bigoplus_{v\in V} C_v$ with the ℓ^1 product norm given by $\|(x_v)\|_1 := \sum_{v\in V} \|x_v\|_v$ induces a norm $\|\cdot\|$ on its quotient C_E via $\|x\| := \inf_{x=\pi(y)} \|y\|_1$.

Corollary 6.7. *With the notation above, the two norms $\|\cdot\|$ and $\|\cdot\|_E$ on C_E agree.*

Proof. This is simply an equivalent statement of Theorem 6.2 for chains in edge groups. □

6.1. **Scl in the edge group of amalgamated free products.** Corollary 6.7 is particularly simple in the special case of amalgams $G = A \star_C B$ so that the unit norm ball has a precise description. To do so, we need the following notion from convex analysis. The *algebraic closure* of a set A consists of points x such that there is some $v \in V$ so that for any $\epsilon > 0$, there is some $t \in [0, \epsilon]$ with $x + tv \in A$. If A is convex, the algebraic closure of A coincides with $\mathrm{lin}(A)$ defined in [Hol75, p. 9], which is also convex. If A in addition finite-dimensional, then its algebraic closure agrees with the topological closure of A [Hol75, p. 59].

Theorem 6.8. *Let $G = A \star_C B$ be the amalgamated free product associated to inclusions $\iota_A : C \to A$ and $\iota_B : C \to B$. Then for any chain $c \in C_1^H(C)$, we have*

$$\mathrm{scl}_G(c) = \inf_{\substack{c_1, c_2 \in C_1^H(C) \\ c_1 + c_2 = c}} \{\mathrm{scl}_A(c_1) + \mathrm{scl}_B(c_2)\}. \tag{6.2}$$

The unit ball of scl *on C equals the algebraic closure of* $\mathrm{conv}(B_A \cup B_B)$*, where* $\mathrm{conv}(\cdot)$ *denotes the convex hull, B_A and B_B are the unit norm balls of the pullbacks of* scl_A *and* scl_B *via ι_A and*

ι_B *on* $C_1^H(C)$ *respectively. If* $C \cong \mathbb{Z}$ *with generator* t*, then*

$$\mathrm{scl}_G(t) = \min\{\mathrm{scl}_A(t), \mathrm{scl}_B(t)\}.$$

Proof. Equation (6.2) is an explicit equivalent statement of Corollary 6.7 in our case. The assertion on the unit norm ball is an immediate consequence of (6.2) and Lemma 6.12 that we will prove below. When $C \cong \mathbb{Z} = \langle t \rangle$ and $c = t$, we have $c_1 = \lambda t$ and $c_2 = (1-\lambda)t$ for some $\lambda \in \mathbb{R}$ and $\mathrm{scl}_G(t) = \inf_\lambda\{|\lambda|\mathrm{scl}_A(t) + |1-\lambda|\mathrm{scl}_B(t)\}$, where the optimization is achieved at either $\lambda = 0$ or $\lambda = 1$. □

In the special case where the edge group is $\mathbb{Z}$, we can construct extremal quasimorphisms for edge group elements, which also gives a different way to establish the formula.

Proposition 6.9. *Let* $G = A \star_C B$*, where* $C \cong \mathbb{Z}$ *and is generated by* t*. Then we can construct the an extremal quasimorphisms for* t *on* G *in terms of the extremal quasimorphism for* t *on* A *and* B*.*

Proof. Let $\phi_A\colon A \to \mathbb{R}$ and $\phi_B\colon B \to \mathbb{R}$ be quasimorphisms which are extremal for t in A and B respectively in the sense of Corollary 3.9. Without loss of generality assume that $\mathrm{scl}_A(t) \le \mathrm{scl}_B(t)$. By possibly rescaling ϕ_A and ϕ_B we may assume that $\phi_A(t) = \phi_B(t) = \mathrm{scl}_A(t)$ and thus $D(\phi_A) = \frac{1}{4}$ and $D(\phi_B) = \frac{\mathrm{scl}_A(t)}{4\mathrm{scl}_B(t)} \le \frac{1}{4}$.

We define $\phi\colon G \to \mathbb{R}$ as follows. Every element $h \in G$ may be written as a reduced word

$$h = a_1 b_1 \cdots a_n b_n,$$

where $a_i \in A \setminus C$ and $b_i \in B \setminus C$ apart from possibly a_1 and b_n. For any other reduced form

$$h = a_1' b_1' \cdots a_{n'}' b_{n'}'$$

it is well known that $n = n'$ and that there are elements $c_i, d_i \in C$ such that $a_i' = c_{i-1}^{-1} a_i d_i$ and $b_i' = d_i^{-1} b_i c_i$. We set

$$\phi(h) = \sum_{i=1}^{n} \phi_A(a_i) + \phi_B(b_i).$$

By our choice of ϕ_A, we have $\phi_A(a_i') = -\phi_A(c_{i-1}) + \phi_A(a_i) + \phi_A(d_i)$ and similarly for ϕ_B. Thus ϕ is well defined, independent of the choices for a_i and b_i.

Next we compute the defect of ϕ. Let $h_1, h_2 \in G$ be two nontrivial elements. Then using the normal form above we see that

$$\begin{aligned} h_1 &= c_1 x_1 c_2^{-1} \\ h_2 &= c_2 x_2 c_3^{-1} \\ (h_1 h_2)^{-1} &= c_3 x_3 c_1^{-1}, \end{aligned}$$

where either

(1) $x_1, x_2, x_3 \in A$ such that $x_1 x_2 x_3 = 1$ and $c_1, c_2, c_3 \in G$ where every c_i is either trivial or ends with an element in $B \setminus C$ in the above normal form, or
(2) $x_1, x_2, x_3 \in B$ such that $x_1 x_2 x_3 = 1$ and $c_1, c_2, c_3 \in G$ where every c_i is either trivial or ends with an element in $A \setminus C$ in the above normal form.

In both cases we see that

$$\begin{aligned} \phi(g) + \phi(h) - \phi(gh) &= \phi(g) + \phi(h) + \phi((gh)^{-1}) \\ &= \phi(c_1) + \phi(x_1) - \phi(c_2) + \phi(c_2) + \phi(x_2) - \phi(c_3) + \phi(c_3) + \phi(x_3) - \phi(c_1) \\ &= \phi(x_1) + \phi(x_2) - \phi(x_1 x_2), \end{aligned}$$

and hence we see that

$$|\phi(g) + \phi(h) - \phi(gh)| = |\phi_A(x_1) + \phi_A(x_2) - \phi_A(x_1x_2)| \leq D(\phi_A),$$

if $x_1, x_2, x_3 \in A$ and

$$|\phi(g) + \phi(h) - \phi(gh)| = |\phi_B(x_1) + \phi_B(x_2) - \phi_B(x_1x_2)| \leq D(\phi_B),$$

Thus $D(\phi) \leq 1/4$ by the above assumption on the defects of ϕ_A and ϕ_B. By Proposition 3.1, we have that $D(\bar{\phi}) \leq 2D(\phi)$ and thus $D(\bar{\phi}) \leq 1/2$. Moreover, it is clear that $\bar{\phi}(t) = \mathrm{scl}_A(t)$. Thus,

$$\frac{\bar{\phi}(t)}{2D(\phi)} = \mathrm{scl}_A(t) = \mathrm{scl}_G(t)$$

by Theorem 6.8 and thus $\bar{\phi}$ is extremal. $\square$

The simple formula (6.2) in the case of amalgams allows us to describe the unit norm ball of $(\mathrm{scl}_G)|_{C_1^H(C)}$. To accomplish this, we introduce the following definition.

Definition 6.10. For two degenerate norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space V. The ℓ^1-*mixture norm* $\|\cdot\|_m$ is defined as

$$\|v\|_m = \inf_{v_1+v_2=v}(\|v_1\|_1 + \|v_2\|_2).$$

Then the norm $(\mathrm{scl}_G)|_{C_1^H(C)}$ is the ℓ^1-mixture of scl_A and scl_B by (6.2) in Theorem 6.8.

Let V_i^f, V_i^z and B_i be the domain, vanishing locus and unit ball of the norm $\|\cdot\|_i$ for $i = 1, 2$ respectively. Note that the domain V_m^f of $\|\cdot\|_m$ is $V_1^f + V_2^f$, and the vanishing locus V_m^z of $\|\cdot\|_m$ contains $V_1^z + V_2^z$ as a subspace.

Lemma 6.11. *The vanishing locus V_m^z of $\|\cdot\|_m$ is $V_1^z + V_2^z$ if V is finite-dimensional.*

Proof. Fix an arbitrary genuine norm $\|\cdot\|$ on V. Let E_1 be a subspace of V_1^f such that V_1^f is the direct sum of E_1 and V_1^z. Then $\|\cdot\|_1$ is a genuine norm on E_1, a finite-dimensional space, and thus there exists $r_1 > 0$ such that any $u \in E_1$ with $\|u\|_1 \leq 1$ has $\|u\| \leq r_1$. It follows that every vector v in B_1 can be written as $v_0 + u$ where $v_0 \in V_1^z$ and $\|u\| \leq r_1$. A similar result holds for $\|\cdot\|_2$ with some constant $r_2 > 0$. Then for any $v \in V_m^z$, for any $\epsilon > 0$, we have $v = v_1 + v_2 + u_1 + u_2$ for some $v_i \in V_i^z$ and u_i satisfying $\|u_i\| \leq \epsilon r_i$, $i = 1, 2$. Hence V_m^z is contained in the closure of $V_1^z + V_2^z$. But $V_1^z + V_2^z$ is already closed since V is finite-dimensional. $\square$

The unit norm ball B_m of an ℓ^1-mixture norm $\|\cdot\|_m$ has a simple description.

Lemma 6.12. *The unit norm ball B_m is the algebraic closure of* $\mathrm{conv}(B_1 \cup B_2)$*, where* $\mathrm{conv}(\cdot)$ *takes the convex hull. If the underlying space is finite-dimensional, then we can take the topological closure of* $\mathrm{conv}(B_1 \cup B_2)$ *instead.*

Proof. Fix any $v \in B_m$. For any $\epsilon > 0$, there exist v_1, v_2 with $v = v_1 + v_2$ and $\|v_1\|_1 + \|v_2\|_2 < 1 + \epsilon$. Let $u_i \in B_i$ be $v_i/\|v_i\|_i$ if $\|v_i\|_i \neq 0$, and 0 otherwise. With $d = \max(1, \|v_1\|_1 + \|v_2\|_2)$, we have

$$\frac{v}{d} = \frac{\|v_1\|_1}{d} \cdot u_1 + \frac{\|v_2\|_2}{d} \cdot u_2 + (1 - \frac{\|v_1\|_1 + \|v_2\|_2}{d}) \cdot 0 \in \mathrm{conv}(B_1 \cup B_2).$$

It follows that for any $\epsilon > 0$, there is some $0 \leq t < \epsilon$ such that $(1-t)v \in \mathrm{conv}(B_1 \cup B_2)$. Thus v is in the algebraic closure of $\mathrm{conv}(B_1 \cup B_2)$.

Conversely, fix any v in the algebraic closure of $\mathrm{conv}(B_1 \cup B_2)$. Note that $\mathrm{conv}(B_1 \cup B_2)$ is a subset of $V_1^f + V_2^f$ and that any linear subspace is algebraically closed, so the algebraic closure of $\mathrm{conv}(B_1 \cup B_2)$ is a subset of $V_1^f + V_2^f$. Then by definition, there is some $u = u_1 + u_2$ with

$u_i \in V_i^f$ such that for any $\epsilon > 0$, we have $v + tu \in \mathrm{conv}(B_1 \cup B_2)$ for some $0 \le t \le \epsilon$. Thus $v = \lambda v_1 + (1-\lambda)v_2 - tu = [\lambda v_1 - tu_1] + [(1-\lambda)v_2 - tu_2]$ for some $\lambda \in [0,1]$ and $v_i \in B_i$. We see

$$\begin{aligned} \|v\|_m &\le \|\lambda v_1 - tu_1\|_1 + \|(1-\lambda)v_2 - tu_2\|_2 \\ &\le \lambda\|v_1\|_1 + (1-\lambda)\|v_2\|_2 + t(\|u_1\|_1 + \|u_2\|_2) \\ &\le 1 + \epsilon(\|u_1\|_1 + \|u_2\|_2). \end{aligned}$$

Since ϵ is arbitrary and $\|u_1\|_1 + \|u_2\|_2$ is finite, we get $v \in B_m$. □

Remark 6.13. It is necessary to take the algebraic closure. On $\mathbb{R}^2 = \{(x,y)\}$, let $\|(x,y)\|_1 = \infty$ if $y \neq 0$ and $\|(x,0)\|_1 = |x|$, and let $\|(x,y)\|_2 = \infty$ if $x \neq 0$ and $\|(0,y)\|_2 = 0$. Then their ℓ^1-mixture has formula $\|(x,y)\|_m = |x|$. Thus the unit balls $B_1 = [-1,1] \times \{0\}$, $B_2 = \{0\} \times \mathbb{R}$ and $B_m = [-1,1] \times \mathbb{R}$. Thus $\mathrm{conv}(B_1 \cup B_2) = (-1,1) \times \mathbb{R} \cup \{(\pm 1, 0)\}$ does not agree with B_m but its algebraic closure does.

Lemma 6.12 confirms the assertion on the unit norm ball in Theorem 6.8 and finishes its proof. This allows us to look at explicit examples showing how scl behaves under surgeries.

Example 6.14. Let Σ be a once-punctured torus with boundary loop γ. Then $X_A := S^1 \times \Sigma$ is a compact 3-manifolds with torus boundary T_A. Let γ_A be a chosen section of γ in T_A and let τ_A be a simple closed curve on T_A representing the S_1 factor. Let $C := \pi_1(T_A) = \langle \gamma_A, \tau_A \rangle \cong \mathbb{Z}^2$ be the peripheral subgroup of $A := \pi_1(X_A)$, where we abuse the notation and use γ_A, τ_A to denote their corresponding elements in $\pi_1(T_A)$.

Let X_B be another copy of X_A, where T_B, γ_B and τ_B correspond to T_A, γ_A and τ_A respectively. For any coprime integers p, q, there is an orientable closed 3-manifold $M_{p,q}$ (not unique) obtained by gluing T_A and T_B via a map $\phi : \pi_1(T_B) \to \pi_1(T_A) = C$ taking γ_B to $p\gamma_A + q\tau_A$. Then $\pi_1(M_{p,q})$ is an amalgam $A \star_C B$ where $B := \pi_1(X_B)$ and the inclusion $C \to B$ is given by $C \xrightarrow{\phi^{-1}} \pi_1(T_B) \hookrightarrow B$.

Identify $H_1(C;\mathbb{R})$ with $\mathbb{R}^2$ with $(1,0)$ representing $[\gamma_A]$ and $(0,1)$ representing $[\tau_A]$. According to Remark 6.3, scl_A and scl_B induce norms on $H_1(C;\mathbb{R}) \cong C_1^H(C)/B_1^H(C)$. Then the norm scl_A on $H_1(C;\mathbb{R})$ has an one-dimensional unit norm ball, which is the segment connecting $(-2,0)$ and $(2,0)$ since $\mathrm{scl}_A(\gamma_A) = \mathrm{scl}_\Sigma(\gamma) = 1/2$. Similarly the unit norm ball of scl_B on $H_1(C;\mathbb{R})$ is the segment connecting $(2p, 2q)$ and $(-2p,-2q)$. By Theorem 6.8, the unit norm ball of $\mathrm{scl}_{M_{p,q}}$ on $H_1(C;\mathbb{R})$ is the convex hull of $\{(\pm 2, 0), \pm(2p, 2q)\}$ (which is already closed), which intersects the positive y-axis at $(0, \frac{q}{p+1})$ when $p, q > 0$. In this case, we have $\mathrm{scl}_{M_{p,q}}(\tau_A) = \frac{p+1}{q}$.

7. Spectral gaps of graph products

In this section we apply Theorem 5.9 to obtain sharp gaps of scl in graph products, which are groups obtained from given collections of groups generalizing both free products and direct products.

Definition 7.1. Let Γ be a *simple* graph (not necessarily connected or finite) and let $\{G_v\}$ be a collection of groups each associated to a vertex of Γ. The *graph products* G_Γ is the quotient of the free product $\star G_v$ by the set of relations $\{[g_u, g_v] = 1 \mid g_u \in G_u, g_v \in G_v, u, v \text{ are adjacent}\}$.

Example 7.2. Here are some well known examples.

1. If Γ has no edges at all, then G_Γ is the free product $\star_v G_v$;
2. If Γ is a complete graph, then G_Γ is the direct product $\oplus_v G_v$;
3. If each $G_v \cong \mathbb{Z}$, then G_Γ is called the *right-angled Artin group* (RAAG for short) associated to Γ;
4. If each $G_v \cong \mathbb{Z}/2\mathbb{Z}$, then G_Γ is called the *right-angled Coxeter group* associated to Γ.

We first introduce some terminologies necessary for the statements and proofs. Denote the vertex set of Γ by $V(\Gamma)$. For any $V' \subset V(\Gamma)$, the *full subgraph* on V' is the subgraph of Γ whose vertex set is V' and edge set consists of all edges of Γ connecting vertices in V'. Any full subgraph Λ gives a graph product denoted G_Λ which is naturally a subgroup of G_Γ. It is actually a retract of G_Γ, by trivializing G_v for all $v \notin \Lambda$. Denote by $lk(v)$ the *link* of a vertex v, which is the full subgraph of $\{w \mid w \text{ is adjacent to } v\}$. The *star* $st(v)$ is the full subgraph of $\{v\} \cup \{w \mid w \text{ is adjacent to } v\}$.

Finally, each element $g \in G_\Gamma$ can be written as a product $g_1 \cdots g_m$ with $g_i \in G_{v_i}$. Such a product is *reduced* if

(1) $g_i \neq id$ for all i, and
(2) $v_i \neq v_j$ whenever we have $i \leq k < j$ such that $[g_i, g_t] = id$ for all $i \leq t \leq k$ and $[g_t, g_j] = id$ for all $k+1 \leq t \leq j$.

It is known that every nontrivial element of G_Γ can be written in a reduced form, which is unique up to certain operations (syllable shuffling) [Gre90, Theorem 3.9]. In particular, any g expressed in the reduced form above is nontrivial in G_Γ.

Lemma 7.3. *Let G_Γ be a graph product. Suppose $g = g_1 \cdots g_m \in G_\Gamma$ ($m \geq 1$) is in reduced form such that for some $n \geq 3$ each $g_i \in G_{v_i}$ has order at least n, $1 \leq i \leq m$. Then g is n-RTF in (G_Γ, G_Λ) for a full subgraph $\Lambda \subset \Gamma$ unless $v_i \in \Lambda$ for all $1 \leq i \leq m$.*

Proof. We proceed by induction on m. The base case $m = 1$ is obvious using the retract from G_Γ to G_{v_1}. For the inductive step, we show g is n-RTF in (G_Γ, G_Λ) if $v_1 \notin \Lambda$, and the other cases are similar. It suffices to prove that g is n-RTF in $(G_\Gamma, G_{\Lambda_1})$ where Λ_1 is the full subgraph of the complement of v_1 in Γ since $G_\Lambda \leq G_{\Lambda_1}$ and using Lemma 5.12 (2). Consider G_Γ as an amalgam $A \star_C B$ with $A = G_{st(v_1)}$, $C = G_{lk(v_1)}$ and $B = G_{\Lambda_1}$. Then there is a unique decomposition of g into $g = a_1 b_1 \cdots a_\ell b_\ell$ with $a_i \in A$ and $b_i \in B$, where each a_i is a maximal subword of $g_1 \cdots g_m$ that stays in $A - C$. To be precise, there is some $\ell \geq 1$ and indices $0 = \beta_0 < \alpha_1 < \beta_1 < \cdots < \alpha_\ell \leq \beta_\ell \leq m$, such that $g = a_1 b_1 \cdots a_\ell b_\ell$, where $a_i = g_{\beta_{i-1}+1} \cdots g_{\alpha_i} \in A$ and $b_i = g_{\alpha_i+1} \cdots g_{\beta_i} \in B$ for $1 \leq i \leq \ell$, and such that

(1) $b_\ell = id$ if $\alpha_\ell = m$;
(2) For each $1 \leq i \leq \ell$, we have $v_j \in st(v_1)$ for all $\beta_{i-1} + 1 \leq j \leq \alpha_i$, and $v_j = v_1$ for some $\beta_{i-1} + 1 \leq j \leq \alpha_i$; and
(3) For each $1 \leq i \leq \ell$ (or $i < \ell$ if $\alpha_\ell = m$), we have $v_j \neq v_1$ for all $\alpha_i + 1 \leq j \leq \beta_i$, and $v_{\alpha_i+1}, v_{\beta_i} \notin st(v_1)$.

Since g is reduced, so are each a_i and b_i. Thus each a_i (resp. b_i, except the case $b_\ell = id$) is n-RTF in (A, C) (resp. (B, C)) by the induction hypothesis, and thus g is n-RTF in (G_Γ, B) by Lemma 5.16, unless $\ell = 1$ and $b_\ell = id$. In the exceptional case we have $v_i \in st(v_1)$ for all $1 \leq i \leq m$, and the assertion is obvious using the direct product structure of $G_{st(v_1)}$ and the fact that $g = g_1 \cdots g_m$ is reduced. $\square$

Theorem 7.4. *Let G_Γ be a graph product. Suppose $g = g_1 \cdots g_m \in G_\Gamma$ ($m \geq 1$) is in cyclically reduced form such that for some $n \geq 3$ each $g_i \in G_{v_i}$ has order at least n, $1 \leq i \leq m$. Then either*

$$\mathrm{scl}_{G_\Gamma}(g) \geq \frac{1}{2} - \frac{1}{n},$$

or the full subgraph Λ on $\{v_1, \ldots, v_m\}$ in Γ is a complete graph. In the latter case, we have

$$\mathrm{scl}_{G_\Gamma}(g) = \mathrm{scl}_{G_\Lambda}(g) = \max \mathrm{scl}_{G_i}(g_i).$$

Proof. Fix any v_k, similar to the proof of Lemma 7.3, we express G_Γ as an amalgam $A \star_C B$, where A, C and B are the graph products associated to $st(v_k)$, $lk(v_k)$ and the full subgraph on $V(\Gamma) - \{v_k\}$ respectively. If there is some $v_i \notin st(v_k)$, then up to a cyclic conjugation, $g = a_1 b_1 \cdots a_\ell b_\ell$ where $\ell \geq 1$, each a_i and b_i is a product of consecutive g_j's such that $b_i \in B - C$ and each $a_i \in A - C$ is of maximal length. Since g is cyclically reduced, each a_i (resp. b_i) is n-RTF in (A, C) (resp. (B, C)) by Lemma 7.3. It follows from Theorem 5.8 that $\mathrm{scl}_{G_\Gamma}(g) \geq 1/2 - 1/n$.

Therefore, the argument above implies $\mathrm{scl}_{G_\Gamma}(g) \geq 1/2 - 1/n$ unless $v_i \in st(v_k)$ for all i, which holds for all k only when the full subgraph Λ on $\{v_1, \ldots, v_m\}$ in Γ is complete. In this case, G_Γ retracts to $G_\Lambda = \oplus G_{v_i}$. Then $v_i \neq v_j$ whenever $i \neq j$ since g is reduced, thus the conclusion follows from Lemma 2.3 (4). $\square$

Remark 7.5. The estimate is sharp in the following sense. For any $g_v \in G_v$ of order $n \geq 2$ and any $g_u \in G_u$ of order $m \geq 2$ with u not equal or adjacent to v, then the retract from G_Γ to $G_u \star G_v$ gives

$$\mathrm{scl}_{G_\Gamma}([g_u, g_v]) = \mathrm{scl}_{G_u \star G_v}([g_u, g_v]) = \frac{1}{2} - \frac{1}{\min(m, n)}$$

by [Che18a, Proposition 5.6].

Theorem 7.6. *Let G_Γ be the graph product of $\{G_v\}$. Suppose for some $n \geq 3$ and $C > 0$, each G_v has no k-torsion for all $2 \leq k \leq n$ and has strong gap C. Then G_Γ has strong gap $\min\{C, 1/2 - 1/n\}$.*

Proof. For any nontrivial $g \in G_\Gamma$ written in reduced form, by Theorem 7.4, we either have $\mathrm{scl}_{G_\Gamma}(g) \geq 1/2 - 1/n$ or $\mathrm{scl}_{G_\Gamma}(g) = \max \mathrm{scl}_{G_i}(g_i) \geq C$. $\square$

Corollary 7.7. *For $n \geq 3$, any graph product of abelian groups without k-torsion for all $2 \leq k \leq n$ have strong gap $1/2 - 1/n$. In particular, all right-angled Artin groups have strong gap $1/2$.*

Unfortunately, our result does not say much about the interesting case of right-angled Coxeter groups due to the presence of 2-torsion.

Question 7.8. Is there a spectral gap for every right-angled Coxeter group? If so, is there a uniform gap?

8. Spectral gap of 3-manifold groups

In this section, we show that any closed connected orientable 3-manifold has a scl spectral gap. All 3-manifolds in this section are assumed to be *orientable*, *connected* and *closed* unless stated otherwise. Throughout this section, we will use scl_M to denote $\mathrm{scl}_{\pi_1(M)}$ and use $\mathrm{scl}_{(M, \partial M)}$ to denote scl in $\pi_1(M)$ relative to the peripheral subgroups when M potentially has boundary.

8.1. **Decompositions of 3-manifolds.** We first recall some important decompositions of 3-manifolds and the geometrization theorem. Every 3-manifold has a unique *prime decomposition* as a connected sum of finitely many prime 3-manifolds. We call a prime 3-manifold *non-geometric* if it does not admit one of the eight model geometries.

By the geometrization theorem, conjectured by Thurston [Thu82] and proved by Perelman [Per02, Per03b, Per03a], the *JSJ decomposition* [JS79, Joh79] cuts each non-geometric prime 3-manifold along a *nonempty* minimal finite collection of embedded disjoint incompressible 2-tori into compact manifolds (with incompressible tori boundary) that each either admits one of the eight model geometries in the interior of *finite volume* or is the twisted I-bundle K over the Klein bottle defined below. Here K is $(T^2 \times [0, 1])/\sigma$, where the involution σ is defined as $\sigma(x, t) = (\tau(x), 1 - t)$ and τ is the unique nontrivial deck transformation for the 2-fold covering of the Klein bottle by T^2. The compact manifold (with torus boundary) K is also recognized as the compact regular neighborhood of any embedded one-sided Klein bottle in any 3-manifold.

We will refer to such a decomposition of prime manifolds as the *geometric decomposition.* The only place that it differs from applying the JSJ decomposition to all prime factors is that some geometric prime 3-manifolds (e.g. of *Sol* geometry) admit nontrivial JSJ decomposition, but we do not cut them into smaller pieces in the geometric decomposition.

Furthermore, five ($\mathbb{S}^3, \mathbb{E}^3, \mathbb{S}^2 \times \mathbb{E}$, Nil, Sol) out of the eight geometries have no non-cocompact lattice [Thu97, Theorem 4.7.10]. It follows that pieces obtained by a nontrivial geometric decomposition are either K or have one of the other three geometries in the interior: $\mathbb{H}^3$, $\mathbb{H}^2 \times \mathbb{E}$ or $\widetilde{\mathrm{PSL}}_2$.

If a compact manifold M with (possibly empty) tori boundary has $\mathbb{H}^2 \times \mathbb{E}$ or $\widetilde{\mathrm{PSL}}_2$ geometry in the interior with finite volume, then it is *Seifert fibered* over a 2-dimensional compact *orbifold* B whose orbifold Euler characteristic $\chi_o(B) < 0$; see [Sco83, Theorems 4.13 and 4.15] and [Thu97, Corollary 4.7.3]. We will introduce basic properties of Seifert fibered spaces and orbifolds in Subsection 8.2.

We summarize the discussion above as the following theorem.

Theorem 8.1. *There is a unique geometric decomposition of every prime* 3*-manifold* M*, which is trivial if* M *already admits one of the eight geometries and is the JSJ decomposition if* M *is non-geometric. Moreover, if* M *is non-geometric, each piece* N *in its geometric decomposition has incompressible tori boundary and exactly one of the following three cases occurs:*

(1) N *has hyperbolic geometry of finite volume in the interior,*
(2) N *is Seifert fibered over a compact orbifold* B *with* $\chi_o(B) < 0$*, or*
(3) N *is homeomorphic to the twisted* I*-bundle* K *over the Klein bottle.*

The minimality of the JSJ decomposition implies that the fiber directions disagree when two boundary components from Seifert fibered pieces are glued. M *being non-geometric implies that no two pieces homeomorphic to* K *are glued together.*

Since the tori in the geometric decomposition are incompressible, the fundamental group $\pi_1(M)$ of any non-geometric prime manifold M has the structure of a graph of groups. We refer to the corresponding tree that $\pi_1(M)$ acts on as the *JSJ-tree.* By basic hyperbolic geometry and the minimality of JSJ decomposition, it is noticed by Wilton–Zalesskii [WZ10, Lemma 2.4] that the action on the JSJ-tree has a nice *acylindricity* property.

Definition 8.2. For any $K \geq 1$, an action of G on a tree is K*-acylindrical* if the fixed point set of any $g \neq id \in G$ has diameter at most K.

Lemma 8.3 (Wilton–Zalesskii)**.** *The action of any non-geometric prime* 3*-manifold on its JSJ-tree is* 4*-acylindrical.*

8.2. 2**-dimensional orbifolds and Seifert fibered** 3**-manifolds.** For our purpose, a compact 2-dimensional (cone-type) *orbifold* B is a compact possibly nonorientable surface with finitely many so-called *cone points* in its interior. Each cone point has an order $n \geq 2$, meaning that locally it is modeled on the quotient of a round disk by a rotation of angle $2\pi/n$. The orbifold Euler characteristic $\chi_o(B)$ is the Euler characteristic of the surface minus $\sum(1 - 1/n_i)$, where the sum is taken over all cone points and n_i is the corresponding order.

Only those orbifolds B with negative $\chi_o(B)$ will appear in our discussion, all of which have hyperbolic structures of finite volume realizing each boundary as a cusp [DM84]. In this case, B can be thought of as $\mathbb{H}^2/\Gamma$, where Γ is a discrete subgroup of $\mathrm{Isom}(\mathbb{H}^2)$ isomorphic to the *orbifold fundamental group* $\pi_1(B)$. It follows that each element in $\pi_1(B)$ of finite order acts by an elliptic isometry such that its order divides the order of some cone point, an element acts by parabolic isometry if and only if its conjugacy class represents a loop on the boundary, and all other elements act by hyperbolic isometries (possibly composed with a reflection about the translation axis). It also follows that, a finite cover of B is a hyperbolic surface, and thus $\pi_1(B)$ is word-hyperbolic.

For any compact 2-dimensional orbifold B with $\chi_o(B) < 0$, it is known that $H^k(\pi_1(B);\mathbb{R}) = 0$ for all $k > 2$, and that $H^2(\pi_1(B);\mathbb{R}) \neq 0$ if and only if B is closed and orientable, in which case a generator is given by the Euler class $\mathrm{eu}(B)$ associated to the $\pi_1(B)$ action on the circle at infinity of the hyperbolic plane (for any fixed hyperbolic structure on B).

A compact 3-manifold M with (possibly empty) tori boundary is *Seifert fibered* over an orbifold B if there is a projection $p: M \to B$ such that

(1) each fiber $p^{-1}(b)$ is S^1,
(2) it is an S^1 bundle away from the preimage of cone points of B, and
(3) for each cone point b of order n, a neighborhood of $p^{-1}(b)$ is obtained by gluing the bottom of a solid cylinder to its top by a rotation of angle $2m\pi/n$ for some m coprime to n.

A fiber $p^{-1}(b)$ is *regular* if b is not a cone point. If b is a cone point of order n, we say the fiber $p^{-1}(b)$ has *multiplicity* n. When $\chi_o(B) < 0$, M is aspherical and thus $\pi_1(M)$ is torsion-free [Sco83, Lemma 3.1], and moreover $\pi_1(M)$ fits into a short exact sequence

$$1 \to \mathbb{Z} \to \pi_1(M) \xrightarrow{p} \pi_1(B) \to 1, \tag{8.1}$$

where the normal $\mathbb{Z}$ subgroup is generated by *any* regular fiber [Sco83, Lemma 3.2]. This is a central extension if the bundle is orientable, or equivalently B is orientable since M is orientable. Note that ∂M is exactly the preimage of ∂B, and thus the $\mathbb{Z}$ subgroup lies in every peripheral $\mathbb{Z}^2$ subgroup of $\pi_1(M)$ if $\partial M \neq \emptyset$.

We will frequently use the map p and monotonicity of scl in our estimates. In many cases, p actually preserves scl.

Lemma 8.4 (Calegari). *Let M be a 3-manifold Seifert fibered over an orbifold B with $\chi_o(B) < 0$. Let $p: \pi_1(M) \to \pi_1(B)$ be induced by the projection and let $z \in \pi_1(M)$ represent a regular fiber. Then $\mathrm{scl}_M(g) = \mathrm{scl}_B(p(g))$ for all null-homologous $g \in \pi_1(M)$ if $H^2(\pi_1(B);\mathbb{R}) = 0$.*

Proof. This is [Cal09b, Proposition 4.30] applied to the exact sequence (8.1). □

When the base space B is closed and orientable, we need some other tools.

Lemma 8.5. *Let M be a closed 3-manifold Seifert fibered over an orientable orbifold B with $\chi_o(B) < 0$. Let $p: \pi_1(M) \to \pi_1(B)$ be induced by the projection and let $z \in \pi_1(M)$ represent a regular fiber. Then $p^*: H^2(\pi_1(B);\mathbb{R}) \to H^2(M;\mathbb{R})$ vanishes if $[z] = 0 \in H_1(M;\mathbb{R})$.*

Proof. Fix a hyperbolic structure on B. Recall that the Euler class $\mathrm{eu}(B)$ generates $H^2(\pi_1(B);\mathbb{R})$. If $[z] = 0$, then $p^*: H^1(\pi_1(B);\mathbb{R}) \to H^1(M;\mathbb{R})$ is an isomorphism by the exact sequence (8.1). For any $\sigma \in H^1(M;\mathbb{R})$, there is some $\sigma(B) \in H^1(\pi_1(B);\mathbb{R})$ such that $\sigma = p^*\sigma(B)$. Since $H^3(\pi_1(B);\mathbb{R}) = 0$, we have

$$\sigma \cup p^*\mathrm{eu}(B) = p^*(\sigma(B) \cup \mathrm{eu}(B)) = 0$$

for all $\sigma \in H^1(M;\mathbb{R})$. Therefore $p^*\mathrm{eu}(B) = 0$ by the Poincaré duality. □

Lemma 8.6. *Suppose the Euler class* eu *associated to an action $\rho: G \to \mathrm{Homeo}^+(S^1)$ is an n-torsion element in $H^2(G;\mathbb{Z})$, then there is a subgroup $G_0 \leq G$ of index n such that the restricted action of G_0 lifts to an action on $\mathbb{R}$ with rotation quasimorphism* rot. *Then $\phi(g) := \mathrm{rot}(g^n)/n$ defines a homogeneous quasimorphism on G of defect $D(\phi) \geq D(\mathrm{rot})$ such that the bounded Euler class $\mathrm{eu}_b = \delta\phi$.*

Proof. Associated to the short exact sequence of trivial modules $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$, the long exact sequence of cohomology gives

$$H^1(G;\mathbb{R}/\mathbb{Z}) \xrightarrow{d} H^2(G;\mathbb{Z}) \to H^2(G;\mathbb{R}).$$

Since eu has order n, there is a homomorphism $f \in H^1(G;\mathbb{R}/\mathbb{Z})$ such that $df = \mathrm{eu}$, which can be chosen to have order n as well. Thus $f(G) = (\frac{1}{n}\mathbb{Z})/\mathbb{Z} \cong \mathbb{Z}/n$. Let $G_0 := \ker f$ and let $i : G_0 \to G$ be the inclusion. Then the Euler class associated to the induced action by G_0 is $i^*\mathrm{eu} = di^*f = 0$. Hence the action of G_0 lifts to an action on $\mathbb{R}$, and the rotation quasimorphism $\mathrm{rot} \in \mathcal{Q}(G_0)$ satisfies $\delta\mathrm{rot} = i^*\mathrm{eu}_b$, where eu_b is the bounded Euler class associated to the action ρ. Here the map δ is shown in the following diagram with exact rows and columns, where all cohomology groups have $\mathbb{R}$ coefficients.

$$\begin{tikzcd}
 & \mathcal{Q}(\mathbb{Z}/n) = 0 \arrow[d] & H^2_b(\mathbb{Z}/n) = 0 \arrow[d] & \\
H^1(G) \arrow[r] \arrow[d, "i^*", "\cong"'] & \mathcal{Q}(G) \arrow[r, "\delta"] \arrow[d, "i^*"] & H^2_b(\pi_1 G) \arrow[r, "c"] \arrow[d, "i^*"] & H^2(G) \arrow[d, "i^*"] \\
H^1(G_0) \arrow[r] & \mathcal{Q}(G_0) \arrow[r, "\delta"] & H^2_b(\pi_1 G_0) \arrow[r] & H^2(G_0)
\end{tikzcd}$$

The first vertical map is an isomorphism since G_0 has finite index in G. Since the Euler class eu vanishes in $H^2(G;\mathbb{R})$, there is a quasimorphism $\phi \in \mathcal{Q}(G)$ such that $\mathrm{eu}_b = \delta\phi$. A simple diagram chasing shows that ϕ can be chosen so that $i^*\phi = \mathrm{rot}$. Hence $\phi(g) = \frac{1}{n}\mathrm{rot}(g^n)$ for all g and $D(\phi) \geq D(\mathrm{rot})$. ☐

Lemma 8.7. *Let M be a closed 3-manifold Seifert fibered over an orientable orbifold B with $\chi_o(B) < 0$. Let $p : \pi_1(M) \to \pi_1(B)$ be induced by the projection and let $z \in \pi_1(M)$ represent a regular fiber. Suppose $[z] = 0 \in H_1(M;\mathbb{R})$. Then there is $\phi \in \mathcal{Q}(\pi_1(M))$ with the following properties:*

(1) $\phi(z) \neq 0$,
(2) there is some $k \in \mathbb{Z}_+$ such that $\phi(g) \in \frac{1}{k}\mathbb{Z}$ whenever $p(g)$ is of infinite order, and
(3) ϕ and the image of $\mathcal{Q}(\pi_1(B))/H^1(\pi_1(B))$ under p^* span $\mathcal{Q}(\pi_1(M))/H^1(\pi_1(M))$.

Proof. By Lemma 8.5, $p^* : H^2(\pi_1(B);\mathbb{R}) \to H^2(M;\mathbb{R})$ vanishes since $[z] = 0 \in H_1(M;\mathbb{R})$. Fix a hyperbolic structure on B and let $\mathrm{eu}(B)$ (resp. $\mathrm{eu}_b(B)$) be the (resp. bounded) Euler class associate to the action on $\partial\mathbb{H}^2$. Then $p^*\mathrm{eu}(B) = 0 \in H^2(M;\mathbb{R})$, that is, the Euler class associated to the induced $\pi_1(M)$ action on the circle is k-torsion for some k. By Lemma 8.6, we have a quasimorphism $\phi \in \mathcal{Q}(\pi_1(M))$ such that $p^*\mathrm{eu}_b(B) = \delta\phi$. If $p(g)$ is of infinite order, then it acts on $\mathbb{H}^2$ by a hyperbolic or parabolic isometry and thus has fixed points on the circle, and so does $p(g^k)$. Thus by the construction of ϕ, we have $\phi(g) \in \frac{1}{k}\mathbb{Z}$. Moreover, in our situation the manifold M has $\widetilde{\mathrm{PSL}_2}$ geometry, which implies that $\phi(z) = \frac{1}{k}\mathrm{rot}(z^k) \neq 0$.

Finally, similar to the proof of [Cal09b, Proposition 4.30], we have the following diagram with exact rows and columns, where all (bounded) cohomology groups are in $\mathbb{R}$ coefficients:

$$
\begin{tikzcd}
 & & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{Q}(\pi_1 B)/H^1(\pi_1 B) \arrow[r, "\delta"] \arrow[d, "p^*"] & H_b^2(\pi_1 B) \arrow[r, "c"] \arrow[d, "p^*", "\cong"'] & H^2(\pi_1 B) \arrow[d, "p^*"] \\
0 \arrow[r] & \mathcal{Q}(\pi_1 M)/H^1(\pi_1 M) \arrow[r, "\delta"] & H_b^2(\pi_1 M) \arrow[r] \arrow[d] & H^2(M) \\
 & & H_b^2(\mathbb{Z}) = 0 &
\end{tikzcd}
$$

For any $f \in \mathcal{Q}(\pi_1 M)/H^1(\pi_1 M)$, there is some $\sigma \in H_b^2(\pi_1 B)$ such that $p^*\sigma = \delta f$. Since $\mathrm{eu}(B)$ generates $H^2(\pi_1 B)$ and $c(\mathrm{eu}_b(B)) = \mathrm{eu}(B)$, we can write $\sigma = \delta\psi + t\mathrm{eu}_b(B)$ for some $t \in \mathbb{R}$ and $\psi \in \mathcal{Q}(\pi_1 B)/H^1(\pi_1 B)$. Thus

$$\delta(f - p^*\psi - t\phi) = p^*(\delta\psi + t\mathrm{eu}_b(B)) - \delta p^*\psi - tp^*\mathrm{eu}_b(B) = 0.$$

This shows that $f = p^*\psi - t\phi$ and finishes the proof. □

See [Sco83] for a more detailed introduction to orbifolds, Seifert fibered spaces, and their relation to the eight geometries.

8.3. **Gaps from hyperbolicity.** We will need a few tools for our estimates. The first is the following spectral gap theorem of word-hyperbolic groups, which is a corollary of [CF10, Theorem A′].

Theorem 8.8 (Calegari–Fujiwara)**.** *Let G be a δ-hyperbolic group with a finite generating set S. Fix finitely many group elements $\{a_i\}$. Then there is a constant $C = C(\delta, |S|, \{a_i\})$ such that, for any $g \in G$ satisfying*

(1) there is no $n \geq 1$ such that g^n is conjugate to g^{-n} in G; and
(2) there is no $m, n \neq 0$ and index i such that g^n is conjugate to a_i^m,

we have $\mathrm{scl}_{(G,\mathcal{A})}(g) \geq C$, where $\mathcal{A}$ is the collection of cyclic groups $\langle a_i \rangle$.

Proof. Since $\{a_i\}$ is a finite collection, we automatically have a uniform bound T on their translation lengths. Thus this follows from Theorem 2.12 (3). □

Their technique also has applications to groups acting on hyperbolic spaces. The special case of action on trees is carried out carefully by Clay–Forester–Louwsma [CFL16] to make the estimate explicit. We will use the following theorem from [CFL16, Theorem 6.11]. Note the potential confusion that a K-acylindrical action by our definition is $(K+1)$-acylindrical in [CFL16], and the statement below has been tailored for our use.

Theorem 8.9 (Clay–Forester–Louwsma)**.** *Suppose G acts K-acylindrically on a tree and let N be the smallest integer greater than or equal to $(K+3)/2$. If $g \in G$ is hyperbolic, then either $\mathrm{scl}_G(g) \geq 1/12N$ or $\mathrm{scl}_G(g) = 0$. Moreover, the latter case occurs if and only if g is conjugate to g^{-1}.*

The last tool is a strengthened version of Calegari's gap theorem for hyperbolic manifolds [Cal08, Theorem C].

Theorem 8.10 (Calegari)**.** *Let M be a complete hyperbolic manifold of dimension m. Then for any $\kappa > 0$, there is a constant $\delta(\kappa, m) > 0$ such that for any hyperbolic element $\alpha \in \pi_1(M)$ with $\mathrm{scl}_{(M,\partial M)}(a) < \delta$, the unique geodesic loop γ representing α has hyperbolic length no more than κ.*

Proof. The proof of the original theorem [Cal08, Theorem C] written in [Cal09b, Chapter 3] works without much change. We briefly go through it for completeness.

Let S be a relative admissible surface for α of degree n. By simplifying S, we may assume S to be a *pleated surface* and $-\chi(S)/n = -\chi^-(S)/n$. That is, S is a hyperbolic surface of finite volume with geodesic boundary, the map $f : S \to M$ takes cusps into cusps and preserves the lengths of all rectifiable curves, and each point $p \in S$ is in the interior of a straight line segment which is mapped by f to a straight line segment. The nice properties that we will use are $\mathrm{area}(S) = -2\pi\chi(S)$ and f preserves lengths, in particular $\mathrm{length}(\partial S) = n \cdot \mathrm{length}(\gamma)$.

Choose ϵ small compared to the 2-dimensional Margulis constant and $\mathrm{length}(\gamma)$, and take the thin-thick decomposition of DS, where DS is the double of S along its geodesic boundaries. Let DS_{thick} (resp. DS_{thin}) be the part with injectivity radius $\geq 2\epsilon$ (resp. $< 2\epsilon$), and let S_{thick} (resp. S_{thin}) be $S \cap DS_{\mathrm{thick}}$ (resp. $S \cap DS_{\mathrm{thin}}$). Then S_{thin} is a union of cusp neighborhoods, open embedded annuli around short simple geodesic loops, and open embedded rectangles between pairs of geodesic segments of ∂S which are distance $< \epsilon$ apart at every point. Each such a rectangle doubles to an open annulus in DS_{thin}. Let r be the number of rectangles and a be the number of annuli in S_{thin}, then there are $2s + r$ annuli in DS_{thin}, which are disjoint and non-isotopic. Hence $2s + r$ is no more than the maximal number of disjoint non-isotopic simple closed curves on DS, and thus

$$r \leq 2s + r \leq -\frac{3}{2}\chi(DS) = -3\chi(S).$$

For convenience, add the cusp neighborhoods and open annuli of S_{thin} back to S_{thick} so that S_{thin} consists of r thin rectangles. By definition, the $\epsilon/2$-neighborhood of the geodesic boundary in S_{thick} is *embedded*, and thus

$$-2\pi\chi(S) = \mathrm{area}(S) \geq \mathrm{area}(S_{\mathrm{thick}}) \geq \frac{\epsilon}{2}\mathrm{length}(\partial S \cap S_{\mathrm{thick}}).$$

Since each component of S_{thin} intersects ∂S in two components, there are at most $-6\chi(S)$ components of $\partial S \cap S_{\mathrm{thin}}$, while their total length

$$\mathrm{length}(\partial S \cap S_{\mathrm{thin}}) = \mathrm{length}(\partial S) - \mathrm{length}(\partial S \cap S_{\mathrm{thick}}) \geq n \cdot \mathrm{length}(\gamma) - \frac{-4\pi\chi(S)}{\epsilon}.$$

Therefore, at least one component σ of $\partial S \cap S_{\mathrm{thin}}$ satisfies

$$\mathrm{length}(\sigma) \geq \frac{n \cdot \mathrm{length}(\gamma) - \frac{-4\pi\chi(S)}{\epsilon}}{-6\chi(S)} = \frac{n \cdot \mathrm{length}(\gamma)}{-6\chi(S)} - \frac{2\pi}{3\epsilon}.$$

On the other hand, $\mathrm{length}(\sigma)$ cannot be much longer than $\mathrm{length}(\gamma)$, otherwise we will have two long anti-aligned geodesic segments on γ, which according to the Margulis lemma would violate either the discreteness of $\pi_1(M)$ or the fact that there are no order-2 elements in $\pi_1(M)$. More precisely, choosing ϵ small in the beginning so that 4ϵ is less than the m-dimensional Margulis constant, it is shown in the original proof of [Cal09b, Theorem 3.9] that

$$\mathrm{length}(\sigma) \leq 2 \cdot \mathrm{length}(\gamma) + 4\epsilon.$$

Combining this inequality with our earlier estimates, we have

$$2 \cdot \mathrm{length}(\gamma) + 4\epsilon \geq \frac{n \cdot \mathrm{length}(\gamma)}{-6\chi(S)} - \frac{2\pi}{3\epsilon},$$

or equivalently

$$\left(\frac{n}{-6\chi^-(S)} - 2\right)\mathrm{length}(\gamma) \leq 4\epsilon + \frac{2\pi}{3\epsilon}.$$

The result follows since ϵ is a constant depending only on the dimension m. $\square$

8.4. **Estimates of scl in 3-manifolds.** The main theorem of this section is the following gap theorem.

Theorem 8.11. *For any* 3*-manifold* M*, there is a constant* $C = C(M) > 0$ *such that for any* $g \in \pi_1(M)$*, we have either* $\mathrm{scl}_M(g) \geq C$ *or* $\mathrm{scl}_M(g) = 0$*.*

The size of C does depend on M, but we will locate elements with scl less than $1/48$ and classify those with vanishing scl in Theorem 8.28.

As for concrete examples of elements with small scl, one can perform Dehn fillings on a knot complement to produce a sequence of closed hyperbolic 3-manifolds and loops with hyperbolic lengths and stable commutator lengths both converging to 0; see [CF10, Example 2.4]. Here we give a different example among graph manifolds.

Example 8.12. Consider the manifolds $M_{p,q}$ in Example 6.14 with $p = 1$ and $q \in \mathbb{Z}_+$. The image of the loop τ_A has positive scl $2/q$ converging to 0 as q goes to infinity. In this example, the geometric decomposition is obtained by cutting along the image of the torus T_A, where the resulting manifolds X_A and X_B are both trivially Seifert fibered and admit $\mathbb{H}^2 \times \mathbb{E}$ geometry of finite volume in the interior.

By Lemma 2.5, it suffices to estimate scl of loops in each prime factor. We will treat the fundamental group of a prime manifold as a graph of groups with $\mathbb{Z}^2$ edge groups according to the geometric decomposition. We first establish scl gaps of vertex groups relative to adjacent edge groups, which would help us estimate scl in the vertex groups by Lemma 2.7.

Lemma 8.13. *Let* M *be a compact* 3*-manifold with tori boundary (possibly empty). Suppose the interior of* M *is hyperbolic with finite volume. Then* $\pi_1(M)$ *has a strong spectral gap relative to the peripheral subgroups. Moreover,* $\mathrm{scl}_{(M,\partial M)}(g) = 0$ *happens if and only if* g *conjugates into some peripheral subgroup.*

Proof. If g conjugates into some peripheral subgroup, then obviously $\mathrm{scl}_{(M,\partial M)}(g) = 0$. Assume that this is not the case. Let $\kappa(M) > 0$ be a constant so that any geodesic loop in M has length at least κ. Then there is a constant $\delta > 0$ by Theorem 8.10 such that $\mathrm{scl}_{(M,\partial M)}(g) \geq \delta$ for all such g. ◻

A similar result holds for the Seifert fibered manifolds. Here we deal with the case where the manifold has nonempty boundary. See Theorem 8.27 for the case of closed Seifert fibered manifolds.

Lemma 8.14. *Let* M *be a compact* 3*-manifold with* nonempty *tori boundary Seifert fibered over an orbifold* B *such that* $\chi_o(B) < 0$*. Then* $\pi_1(M)$ *has a spectral gap relative to the peripheral* $\mathbb{Z}^2$*-subgroups. Moreover, let* z *represent a regular fiber, then there is a constant* $K_M > 0$ *such that* $\mathrm{scl}_{(M,\partial M)}(g) = 0$ *if and only if one of the following cases occurs:*

(1) g *conjugates into some peripheral subgroup;*
(2) $g^n = z^m$ *for some* $n > 0$ *no exceeding the maximal order of cone points on* B *and some* $m \in \mathbb{Z}$*; or*
(3) $p(g)$ *is conjugate to* $p(g^{-1})$ *in* $\pi_1(B)$ *and* $\mathrm{scl}_M(g^2 - z^m) = 0$ *for some* $m \in \mathbb{Z}$*, where* $p : \pi_1(M) \to \pi_1(B)$ *is the projection map.*

Proof. Since B has nonempty boundary, Lemma 8.4 implies that $H^2(\pi_1(B); \mathbb{R}) = 0$ and p preserves scl . That is, for any $g \in \pi_1(M)$, either $\mathrm{scl}_M(g) = \infty$ or $\mathrm{scl}_M(g) = \mathrm{scl}_B(p(g))$. Let z represent a regular fiber.

If $p(g)$ is of finite order n, then by the exact sequence (8.1), we have $z^m = g^n$ for some integer m, where n divides the order of some cone points on B by the structure of $\pi_1(B)$. We have $\mathrm{scl}_{(M,\partial M)}(g) = 0$ since z is peripheral.

Suppose $p(g)$ is of infinite order in the sequel. As we mentioned earlier, $\pi_1(B)$ is word-hyperbolic and has a discrete faithful action on the hyperbolic plane with finite volume quotient B. Then $p(g)$ either acts as a parabolic isometry or acts as a (possibly orientation-reversing) hyperbolic isometry.

(1) If $p(g)$ is parabolic, then it conjugates into a peripheral $\mathbb{Z}$-subgroup of $\pi_1(B)$, which implies that g conjugates into a peripheral $\mathbb{Z}^2$ subgroup of $\pi_1(M)$ and $\mathrm{scl}_{(M,\partial M)}(g) = 0$.

(2) If $p(g)$ is hyperbolic and $hp(g^n)h^{-1} \neq p(g^{-n})$ for any $n \neq 0$ and $h \in \pi_1(B)$. Fix a generator g_i for each peripheral subgroup of $\pi_1(B)$. Since $p(g)$ acts as a hyperbolic isometry, $hp(g^n)h^{-1} \neq g_i^m$ for any $m, n \neq 0$ and any $h \in \pi_1(B)$. Then Theorem 8.8 implies $\mathrm{scl}_{(B,\partial B)}(p(g)) \geq C$ for a constant $C = C(B)$, where $\mathrm{scl}_{(B,\partial B)}$ denotes scl of $\pi_1(B)$ relative to its peripheral $\mathbb{Z}$ subgroups. This implies $\mathrm{scl}_{(M,\partial M)}(g) \geq \mathrm{scl}_{(B,\partial B)}(p(g)) \geq C$ by monotonicity of relative scl.

(3) If $p(g)$ is hyperbolic and $hp(g^n)h^{-1} = p(g^{-n})$ for some $n \geq 1$ and $h \in \pi_1(B)$. Then h must interchange the two endpoints of the axis of $p(g)$, and thus $hp(g)h^{-1} = p(g^{-1})$. This implies $\tilde{h}g\tilde{h}^{-1}g = z^m$ for some $\tilde{h} \in \pi_1(M)$ and $m \in \mathbb{Z}$. Hence the chain $g - \frac{m}{2}z$ is null-homologous. Since p preserves scl and $p(g)$ is conjugate to its inverse, we have $\mathrm{scl}_M(g - \frac{m}{2}z) = \mathrm{scl}_B(p(g)) = 0$. Since z is peripheral, we conclude that $\mathrm{scl}_{(M,\partial M)}(g) = 0$ in this case.

Combining the cases above, any $g \in \pi_1(M)$ either has $\mathrm{scl}_{(M,\partial M)}(g) \geq C$ or $\mathrm{scl}_{(M,\partial M)}(g) = 0$. Moreover, if $\mathrm{scl}_{(M,\partial M)}(g) = 0$, then

(1) either $p(g)$ is of finite order and $g^n \in \langle z \rangle$ for some $n > 0$ no exceeding the maximal order of cone points on B;

(2) or $p(g)$ is of infinite order and
   (a) $p(g)$ is parabolic, which implies that g conjugates into some peripheral subgroup; or
   (b) $p(g)$ is hyperbolic, $p(g)$ is conjugate to $p(g^{-1})$, and $\mathrm{scl}_M(g^2 - z^m) = 0$ for some $m \in \mathbb{Z}$.

□

Remark 8.15. In the proof above, the gap C comes from the spectral gap of scl in 2-orbifolds relative to nonempty boundary, which can be made uniform and explicit with $C = 1/24$.

Next we control scl of integral chains in the edge spaces in the geometric decomposition of a prime manifold (Theorem 8.20). We need the following two lemmas and their corollaries.

Lemma 8.16. *Let M be a compact 3-manifold with boundary consisting of tori T and T_i, $i \in I$ (I could be empty). Suppose the interior of M is hyperbolic with finite volume. Then there exists $C > 0$ such that $\mathrm{scl}_{(M,\{T_i\})}(g) \geq C$ for any $g \neq id \in \pi_1(T)$.*

Proof. By the hyperbolic Dehn filling theorem [Thu78, Theorem 5.8.2], we can fix two different Dehn fillings of the end T such that the resulting manifolds M_1 and M_2 are both hyperbolic with cusp ends T_i, $i \in I$. For any $g \neq id \in \pi_1(T)$, it is a nontrivial hyperbolic element in at least one of $\pi_1(M_1)$ and $\pi_1(M_2)$. Thus the result follows from Lemma 8.13 and monotonicity. □

Corollary 8.17. *Let M be a compact 3-manifold with boundary tori T_i ($i \in I$). Suppose the interior of M is hyperbolic with finite volume. Then for any chain c of the form $\sum_{i\in I} t_i g_i$ with $t_i \in \mathbb{R}$ and $g_i \neq id \in \pi_1(T_i)$, we have $\mathrm{scl}_M(c) > 0$ unless $t_i = 0$ for all $i \in I$.*

Proof. Suppose $t_j \neq 0$ for some j, then Lemma 8.16 provides a constant $C_j > 0$ such that

$$\mathrm{scl}_M(\sum t_i g_i) \geq |t_j| \cdot \mathrm{scl}_{(M,\{T_i\}_{i\neq j})}(g_j) \geq |t_j| C_j > 0.$$

□

Lemma 8.18. *Let M be a compact 3-manifold with boundary consisting of tori T and T_i, $i \in I$ (I could be empty). Suppose M is Seifert fibered with bundle projection $p : M \to B$ where B is an orbifold with $\chi_o(B) < 0$. Then there exists $C > 0$ such that $\mathrm{scl}_{(M,\{T_i\})}(g) \geq C$ for any $g \in \pi_1(T) \setminus \ker p$.*

Proof. We have the short exact sequence (8.1). Consider any $g \in \pi_1(T) \setminus \ker p$. Fix any hyperbolic structure of B realizing boundaries as cusps. Then $p(g)$ is a parabolic element. We know $hp(g^n)h^{-1} \neq p(g^{-n})$ for any $h \in \pi_1(B)$ and any $n \neq 0$ since otherwise h must be a hyperbolic reflection, which cannot appear in $\pi_1(B)$. Since different boundary components of B cannot be homotopic and $\pi_1(B)$ is δ-hyperbolic, Theorem 8.8 implies the existence of $C > 0$ such that

$$\mathrm{scl}_{(M,\{T_i\})}(g) \geq \mathrm{scl}_{(B,\{p(T_i)\})}(p(g)) \geq C$$

for all $g \in \pi_1(T) \setminus \ker p$. $\square$

In the lemma above, one can work out an explicit uniform bound $C = 1/12$ by expressing $\pi_1(B)$ explicitly as a free product of cyclic groups.

Corollary 8.19. *Let M be a compact 3-manifold with nonempty boundary tori T_i ($i \in I$). Suppose M is Seifert fibered with bundle projection $p : M \to B$ where B is an orbifold with $\chi_o(B) < 0$. Then for any chain c of the form $\sum_{i\in I} t_i g_i$ with $t_i \in \mathbb{R}$ not all zero and $g_i \neq id \in \pi_1(T_i)$, we have $\mathrm{scl}_M(c) = 0$ if and only if $g_i \in \ker p$ for all $i \in I$ and $[c] = 0 \in H_1(M, \mathbb{R})$.*

Proof. If $g_j \notin \ker p$ for some j, then Lemma 8.18 provides a constant $C_j > 0$ such that

$$\mathrm{scl}_M(\sum t_i g_i) \geq |t_j| \cdot \mathrm{scl}_{(M,\{T_i\}_{i\neq j})}(g_j) \geq |t_j| C_j > 0.$$

Suppose $g_i \in \ker p$ for all $i \in I$. Note that $H^2(\pi_1(B); \mathbb{R}) = 0$ and p preserves scl by Lemma 8.4 since B has boundary. Hence if $[c] = 0 \in H_1(M, \mathbb{R})$, then $\mathrm{scl}_M(c) = \mathrm{scl}_B(p(c)) = \mathrm{scl}_B(0) = 0$. $\square$

Theorem 8.20. *Let M be a non-geometric prime 3-manifold. Let $\mathcal{T}$ be the collection of tori in the JSJ decomposition of M. Then there is a constant $C_M > 0$ such that, for any integral chain $c = \sum_{T\in\mathcal{T}} g_T$ with $g_T \in \pi_1(T) \cong \mathbb{Z}^2$, we have either $\mathrm{scl}_M(c) = 0$ or $\mathrm{scl}_M(c) \geq C_M$. Moreover, if c is a single loop supported in a JSJ torus T, then $\mathrm{scl}_M(c) = 0$ if and only if one of the following cases occurs:*

(1) T is the boundary of a twisted I bundle K over the Klein bottle such that c is null-homologous in K;

(2) $c = z^m$ for some $m \in \mathbb{Z}$, where z represents a regular fiber in a Seifert fibered piece N so that either the base space B is nonorientable or at least one boundary component of N is glued to a twisted I-bundle K over the Klein bottle in the JSJ decomposition so that the regular fibered is glued to a null-homologous loop in K;

(3) T identifies boundary components $\partial_1 \subset N_1, \partial_2 \subset N_2$ of (possibly the same) pieces N_1, N_2, and $c = ab$, such that either (a, N_1) and (b, N_2) are as in case (1) or (2), or $N_1 = N_2$ is Seifert fibered with regular fiber represented by z so that $a = z^m$ and $b = z^{-m}$ for some $m \in \mathbb{Z}$.

Proof. As we mentioned earlier, the geometric decomposition endows $\pi_1(M)$ with the structure of a graph of groups, where the vertex groups are the fundamental groups of geometric pieces $\mathcal{N}$ and the edge groups are the fundamental groups of those tori $\mathcal{T}$ we cut along.

For each $N \in \mathcal{N}$, let $V_N := \bigoplus_{T\in\partial N} H_1(T; \mathbb{R})$ and equip it with the degenerate norm $\|(h_T)_{T\in\partial N}\|_N := \mathrm{scl}_N(\sum_{T\subset\partial N} h_T)$ where $h_T \in H_1(T)$. Then $\bigoplus_{N\in\mathcal{N}} V_N$ is naturally equipped with $\|\cdot\|_1$, the ℓ^1-product norm of all $\|\cdot\|_N$. As we observed in Section 6, $\bigoplus_{N\in\mathcal{N}} V_N = \bigoplus_{T\in\mathcal{T}} [H_1(T; \mathbb{R}) \oplus H_1(T; \mathbb{R})]$. Let $V_{\mathcal{T}} := \bigoplus_{T\in\mathcal{T}} H_1(T; \mathbb{R})$. Piecing together the addition maps $H_1(T; \mathbb{R}) \oplus H_1(T; \mathbb{R}) \xrightarrow{+} H_1(T; \mathbb{R})$ over all $T \in \mathcal{T}$ gives a projection $\pi : \bigoplus_{N\in\mathcal{N}} V_N \to V_{\mathcal{T}}$, which equips $V_{\mathcal{T}}$ with the quotient norm $\|\cdot\|$ of $\|\cdot\|_1$. By Corollary 6.7 and Remark 6.3, we have $\|(h_T)\| = \mathrm{scl}_M(\sum h_T)$ for any $h_T \in H_1(T)$. If N is the twisted I-bundle over the Klein bottle, then a loop on its boundary has vanishing scl if and only if it is null-homologous since the fundamental group of the Klein bottle is virtually abelian. Combining this with Corollary 8.17 and Corollary 8.19, we note that the vanishing locus of $\|\cdot\|_N$ on V_N is rational for each $N \in \mathcal{N}$. Thus the vanishing locus of $\|\cdot\|_1$ is also rational since it is the direct sum over all N of the vanishing locus of $\|\cdot\|_N$ on V_N. Then its image under the projection π is rational, which is exactly the vanishing locus of $\|\cdot\|$. Hence by Lemma 6.6, the desired constant C_M exists.

Suppose c is a single loop with $\mathrm{scl}_M(c) = 0$. Then there is some $(v_N) \in \bigoplus_{N\in\mathcal{N}} V_N$ such that $\pi(v_n) = c$ and $\|v_N\|_N = 0$ for all N. By Corollary 8.17, we have $v_N = 0$ for all hyperbolic pieces N. By Corollary 8.19, each v_N is a sum of fibers on the boundary components of N with $[v_N] = 0 \in H_1(N;\mathbb{R})$ for all Seifert fibered pieces N. Finally, v_N is a null-homologous loop in N if N is the twisted I-bundle over the Klein bottle. Combining these with the minimality of JSJ decomposition and the fact that M is non-geometric, we obtain the classification of such loops c via a case-by-case study. □

The size of C_M is not explicit in Theorem 8.20. We notice from Example 8.12 that C_M could be very small and depends on how the geometric pieces are glued together.

Theorem 8.21. *Let M be a non-geometric prime 3-manifold. Then for each geometric piece N obtained from the JSJ decomposition of M, there is a constant $C_N > 0$ such that for any g representing a loop in N, we have either $\mathrm{scl}_M(g) = 0$ or $\mathrm{scl}_M(g) \geq C_N$.*

Proof. Endow $\pi_1(M)$ with the structure of a graph of groups from the geometric decomposition, then g conjugates into the vertex group $\pi_1(N)$. The boundary of N consists of a nonempty collection of tori. Let C_M be the bound for integral chains supported in the edge groups from Theorem 8.20. There are three cases:

(1) the interior of N is hyperbolic with finite volume. If g conjugates into a peripheral subgroup of $\pi_1(N)$, then our control on edge groups shows that either $\mathrm{scl}_M(g) = 0$ or $\mathrm{scl}_M(g) \geq C_M$. If g does not conjugate into any peripheral subgroup of $\pi_1(N)$, then by Lemma 5.2 and Lemma 8.13, there exists $C > 0$ such that $\mathrm{scl}_M(g) \geq \mathrm{scl}_{(N,\partial N)}(g) \geq C$ for all such g. Thus the conclusion holds with $C_N := \min(C, C_M)$ in this case.

(2) N is Seifert fibered over an orbifold B such that $\chi_o(B) < 0$. By Lemma 8.14, there is a constant $C = C(N)$ such that either $\mathrm{scl}_{(N,\partial N)}(g) \geq C$ or $\mathrm{scl}_{(N,\partial N)}(g) = 0$. Moreover, $\mathrm{scl}_{(N,\partial N)}(g) = 0$ only occurs in two cases:
   (a) either g^n conjugates into some peripheral subgroup of $\pi_1(N)$ for some $n > 0$ not exceeding the maximal order O_N of cone points on B, then by our control on the edge groups, either $\mathrm{scl}_M(g) = 0$ or $\mathrm{scl}_M(g) \geq C_M/O_N$;
   (b) or $\mathrm{scl}_N(g^2 - z^m) = 0$ for some integer m, where z is a generator of $\ker p \cong \mathbb{Z}$ and $p : \pi_1(N) \to \pi_1(B)$ is the projection map. In this case, by monotonicity, we have $\mathrm{scl}_M(g^2 - z^m) = 0$ and $\mathrm{scl}_M(g) = \frac{|m|}{2}\mathrm{scl}_M(z)$. If $m = 0$ or $\mathrm{scl}_M(z) = 0$, then $\mathrm{scl}_M(g) = 0$; otherwise, $\mathrm{scl}_M(g) \geq \mathrm{scl}_M(z)/2 \geq C_M/2$.

   In summary, we may choose $C_N := \min(C, C_M/O_N, C_M/2)$ in this case.

(3) N is homeomorphic to the regular neighborhood K of a one-sided Klein bottle. Then g^2 is conjugate to an edge group element, and thus either $\mathrm{scl}_M(g) = 0$ or $\mathrm{scl}_M(g) \geq C_N := C_M/2$.

□

Finally, we control scl of hyperbolic elements using Theorem 8.9.

Lemma 8.22. *For any non-geometric prime* 3*-manifold* M *and any* $g \in \pi_1(M)$ *that is hyperbolic for the action on the JSJ-tree, either* $\mathrm{scl}_M(g) \geq 1/48$ *or* g *is conjugate to its inverse, in which case* $\mathrm{scl}_M(g) = 0$*.*

Proof. By Lemma 8.3, the action of $\pi_1(M)$ on the JSJ-tree is 4-acylindrical, thus the result follows from Theorem 8.9 with $K = 4$ and $N = 4$. □

The bound can be improved by Theorem 5.9 if certain pieces does not appear in the geometric decomposition. This is done by verifying the 3-RTF condition using geometry.

Lemma 8.23. *Let* M *be a compact* 3*-manifold with tori boundary and let* T *be a boundary component. Suppose either the interior of* M *is hyperbolic with finite volume, or* M *is Seifert fibered over an orbifold* B *such that* $\chi_o(B) < 0$ *and* B *has no cone points of order* 2*. Then* $\pi_1(T)$ *is* 3*-RTF in* $\pi_1(M)$*.*

Proof. We focus on the case where M is Seifert fibered. We will use the exact sequence (8.1) again. Note that $\pi_1(B)$ acts discretely and faithfully on the hyperbolic plane such that, up to a conjugation, $H := p(\pi_1(T))$ is a subgroup of

$$P := \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} : x \in \mathbb{R} \right\} \cong (\mathbb{R}, +)$$

in $\mathrm{PGL}_2(\mathbb{R}) \cong \mathrm{Isom}(\mathbb{H}^2)$. As a result, each $h \in H$ has a unique square root $\sqrt{h} \in P$, i.e. $(\sqrt{h})^2 = h$. Also note that $P \cap \pi_1(B) = H$.

To show that $\pi_1(T)$ is 3-RTF in $\pi_1(M)$, it suffices to show that H is 3-RTF in $\pi_1(B)$. Suppose $g \in \pi_1(B)$ satisfies $gh_1gh_2 = id$ for some $h_1, h_2 \in H$. We need to show $g \in H$. Let $h^* := \sqrt{h_1h_2} \in P$ and $g^* := gh^* \in \mathrm{Isom}(\mathbb{H}^2)$. Then we have

$$(g^*)^{-1} = (h^*)^{-1}g^{-1} = (h^*)^{-1}h_1gh_2 = ((h^*)^{-1}h_1)g^*((h^*)^{-1}h_1)^{-1},$$

where the last equality uses the fact that P is abelian. We have three cases:

(1) g^* fixes some point in $\mathbb{H}^2$. Then the fixed point set is a geodesic subspace X in $\mathbb{H}^2$, which must be preserved by $(h^*)^{-1}h_1 \in P$. This is impossible unless
   (1a) $(h^*)^{-1}h_1 = id$; or
   (1b) $X = \mathbb{H}^2$.
   In the first subcase, we get $h^* = h_1 = h_2$ and $g^* = (g^*)^{-1}$, but now $g^* = gh^* = gh$ is an element of $\pi_1(B)$, which contains no 2-torsion since B has no order 2 cone points. So $g^* = id$ and $g = h^{-1} \in H$. In the second subcase, we have $g^* = id$, i.e. $g = (h^*)^{-1}$ which lies in $P \cap \pi_1(B) = H$.
(2) g^* is parabolic. Then g^* fixes a unique point on $\partial\mathbb{H}^2$ fixed by $(h^*)^{-1}h_1$. So either
   (2a) $(h^*)^{-1}h_1 = id$; or
   (2b) g^* fixes the unique fixed point of P.
   The first subcase (2a) is similar to (1a). In the second subcase, we have $g^* \in P$, and thus $g = g^*(h^*)^{-1} \in P \cap \pi_1(B) = H$.
(3) g^* is hyperbolic, possibly further composed with a reflection across the axis of translation. Then $(h^*)^{-1}h_1$ must switch the two unique points on $\partial\mathbb{H}^2$ fixed by g^*, which is impossible since $(h^*)^{-1}h_1$ is parabolic.

For the case where the interior of M is hyperbolic with finite volume, a similar argument works by replacing $\mathrm{Isom}(\mathbb{H}^2)$ by $\mathrm{Isom}^+(\mathbb{H}^3)$. This is even easier since M is orientable and $\pi_1(M)$ is torsion-free, and thus we omit the proof.

□

We do not know if the result above is optimal.

Question 8.24. Is there some $n > 3$ such that for any hyperbolic 3-manifold M of finite volume with tori cusps, every peripheral subgroup of $\pi_1(M)$ is n-RTF?

It is essential in Lemma 8.23 to assume that the orbifold B has no cone points of order 2, since otherwise an element representing the fiber over a cone point of order 2 squares to an element representing a regular fiber, which lies in peripheral subgroups. The analogous statement of Lemma 8.23 does not hold for the twisted I-bundle K over a Klein bottle with respect to its torus boundary, since the peripheral $\mathbb{Z}^2$ subgroup has index two in $\pi_1(K)$ which is the fundamental group of the core Klein bottle.

Excluding these pieces in a non-geometric prime 3-manifold, the bound in Lemma 8.22 can be improved to $1/6$.

Lemma 8.25. *Suppose M is a non-geometric prime 3-manifold where none of the pieces in its geometric decomposition is the twisted I-bundle over a Klein bottle or Seifert fibered with a fiber of multiplicity two. Then any $g \in \pi_1(M)$ acting hyperbolically on the JSJ-tree has $\mathrm{scl}_M(g) \ge 1/6$.*

Proof. This is simply a combination of Lemma 8.23 and Theorem 5.9. □

Now we are in a place to prove a gap theorem for prime manifolds. We will first focus on the case of non-geometric prime 3-manifolds.

Theorem 8.26. *Let M be a non-geometric prime 3-manifold. Then there is a constant $C = C(M) > 0$ such that either $\mathrm{scl}_M(g) \ge C$ or $\mathrm{scl}_M(g) = 0$ for any $g \in \pi_1(M)$. Moreover, if $\mathrm{scl}_G(g) < 1/48$, then either g is conjugate to its inverse or it is represented by a loop supported in a single piece of the geometric decomposition of M.*

Proof. Endow $\pi_1(M)$ with the structure of a graph of groups according to the geometric decomposition, where the vertex groups are the fundamental groups of geometric pieces $\mathcal{N}$ and the edge groups are the fundamental groups of those tori $\mathcal{T}$ we cut along. By Lemma 8.22, if g is hyperbolic, then either g is conjugate to its inverse and $\mathrm{scl}_G(g) = 0$, or $\mathrm{scl}_G(g) \ge 1/48$.

On the other hand, by Theorem 8.21, there is a constant $C_N > 0$ for each geometric piece N such that, if g conjugates into $\pi_1(N)$ then either $\mathrm{scl}_M(g) \ge C_N$ or $\mathrm{scl}_M(g) = 0$.

Combining the two parts above, for any $g \in \pi_1(M)$, we have either $\mathrm{scl}_M(g) = 0$ or $\mathrm{scl}_M(g) \ge C$, where $C := \min\{1/48, C_N\} > 0$ with N ranging over all geometric pieces of M. □

For geometric prime manifolds, those with $\mathbb{H}^2 \times \mathbb{E}$ or $\widetilde{\mathrm{PSL}}_2$ geometry need additional attention.

Theorem 8.27. *Let M be a prime 3-manifold with $\mathbb{H}^2 \times \mathbb{E}$ or $\widetilde{\mathrm{PSL}}_2$ geometry. So M is Seifert fibered over some closed orbifold B with $\chi_o(B) < 0$. Then there is a constant $C = C(M) > 0$ such that either $\mathrm{scl}_M(g) \ge C$ or $\mathrm{scl}_M(g) = 0$ for any $g \in \pi_1(M)$. Moreover, $\mathrm{scl}_M(g) = 0$ if and only if*

(1) either B is nonorientable and $g^n h g^n h^{-1}$ represents a multiple of the regular fiber for some $h \in \pi_1(M)$ and $n \in \mathbb{Z}_+$,

(2) or $g^n h g^n h^{-1} = id$ for some $h \in \pi_1(M)$ and $n \in \mathbb{Z}_+$.

Proof. Let $z \in \pi_1(M)$ represent a regular fiber, and let p be the projection in the short exact sequence (8.1). By Theorem 2.12 and the fact that $\pi_1(B)$ is word-hyperbolic, there is some $C(B) > 0$ such that either $\mathrm{scl}_B(\bar{g}) \ge C(B)$ or $\mathrm{scl}_B(\bar{g}) = 0$ for $\bar{g} \in \pi_1(B)$. Hence by monotonicity, $\mathrm{scl}_M(g) \ge \mathrm{scl}_B(p(g)) \ge C(B)$ unless $\mathrm{scl}_B(p(g)) = 0$. Moreover, the latter case occurs if and only if there is some $n \in \mathbb{Z}_+, m \in \mathbb{Z}$ and $h \in \pi_1(M)$ such that $g^n h g^n h^{-1} = z^m$.

If B is nonorientable, then $H^2(\pi_1(B); \mathbb{R}) = 0$ and p preserves scl. Moreover, the bundle must be nonorientable and z is conjugate to z^{-1}. Thus $[g] = m[z]/2n = 0 \in H_1(M; \mathbb{R})$. So we have $\mathrm{scl}_M(g) = 0$ if $\mathrm{scl}_B(p(g)) = 0$.

Suppose B is orientable. Consider two cases:

(1) Suppose $[z] \neq 0 \in H_1(M;\mathbb{R})$. Then $g^n h g^n h^{-1} = z^m$ implies $2n[g] = m[z] \in H_1(M;\mathbb{R})$. If $m = 0$, then g^n is conjugate to g^{-n} and thus $\mathrm{scl}_M(g) = 0$. If $m \neq 0$, then $[g] \neq 0 \in H_1(M;\mathbb{R})$ and thus $\mathrm{scl}_M(g) = \infty$.
(2) Suppose $[z] = 0 \in H_1(M;\mathbb{R})$. Then $p^* : H^2(\pi_1(B);\mathbb{R}) \to H^2(M;\mathbb{R})$ vanishes by Lemma 8.5. Fix a hyperbolic structure on B and let $\mathrm{eu}(B)$ (resp. $\mathrm{eu}_b(B)$) be the (resp. bounded) Euler class associate to the action on $\partial\mathbb{H}^2$. Then $p^*\mathrm{eu}(B) = 0 \in H^2(M;\mathbb{R})$, and Lemma 8.7 implies that $p^*\mathrm{eu}_b(B) = \delta\phi$, where $\phi \in \mathcal{Q}(\pi_1 M)$ and there is some $k \in \mathbb{Z}_+$ such that $\phi(g) \in \frac{1}{k}\mathbb{Z}$ if $p(g)$ is hyperbolic. Moreover, $\mathcal{Q}(\pi_1 M)/H^1(\pi_1 M)$ is spanned by ϕ and the image of $\mathcal{Q}(\pi_1 B)/H^1(\pi_1 B)$ under p^*, and we have $\phi(z) \neq 0$.

   Now consider any $g \in \pi_1(M)$ with $[g] = 0 \in H_1(M;\mathbb{R})$ and $\mathrm{scl}_B(p(g)) = 0$. If $p(g)$ is elliptic, then $g^n = z^m$ for some $m \neq 0$ and $1 \le n \le N$, where N is the largest order of cone points in B. Thus $\mathrm{scl}_M(g) \geq \frac{1}{N}\mathrm{scl}_M(z) > 0$ since $\phi(z) \neq 0$. If $p(g)$ is hyperbolic, then $\phi(g) \in \frac{1}{k}\mathbb{Z}$, which leads to two subcases:
   (a) Suppose $\phi(g) = 0$. Since $\mathrm{scl}_B(p(g)) = 0$, we have $\psi(p(g)) = 0$ for all $\psi \in \mathcal{Q}(\pi_1 B)/H^1(\pi_1 B)$. Thus we have $f(g) = 0$ for all $f \in \mathcal{Q}(\pi_1 M)/H^1(\pi_1 M)$, and thus $\mathrm{scl}_M(g) = 0$ by Bavard's duality. In this case, since $g^n = hg^{-n}h^{-1}z^m$ and z is central, we have $n\phi(g) = \phi(hg^{-n}h^{-1}) + \phi(z^m) = -n\phi(g) + m\phi(z)$. This implies $m = 0$ since $\phi(g) = 0$ and $\phi(z) \neq 0$, thus $g^n h g^n h^{-1} = id$.
   (b) Suppose $\phi(g) \neq 0$, then $\mathrm{scl}_M(g) \geq \frac{|\phi(g)|}{2D(\phi)} \geq \frac{1}{2kD(\phi)}$.

In summary, letting
$$C(M) := \begin{cases} \min\left\{C(B), \frac{\mathrm{scl}_M(z)}{N}, \frac{1}{2kD(\phi)}\right\} & \text{if } B \text{ is orientable and } [z] = 0 \in H_1(M;\mathbb{R}), \\ \min\left\{C(B), \frac{1}{2kD(\phi)}\right\} & \text{otherwise,} \end{cases}$$
we have either $\mathrm{scl}_M(g) \geq C(M)$ or $\mathrm{scl}_M(g) = 0$ for all $g \in \pi_1(M)$. □

Now we prove Theorem 8.11.

Proof of Theorem 8.11. The prime decomposition splits $\pi_1(M)$ as a free product. By Lemma 2.5, we have either $\mathrm{scl}_M(g) \geq 1/12$ or $\mathrm{scl}_M(g) = 0$, unless γ is supported in a single prime factor up to homotopy. Thus it suffices to prove the result for any prime manifold. There are two cases:
(1) If a prime manifold M itself admits one of the eight geometries, then either $\pi_1(M)$ is amenable (actually virtually solvable [Thu97, Theorem 4.7.8]), or M has $\mathbb{H}^3$, $\mathbb{H}^2 \times \mathbb{E}$ or $\widetilde{\mathrm{PSL}}_2$ geometry. In the former case, scl vanishes and the spectral gap property trivially holds. For the latter case, Lemma 8.13 (the empty boundary case) and Theorem 8.27 imply the spectral gap property.
(2) If M is a non-geometric prime manifold, we have proved the result in Theorem 8.26.

□

Moreover, following the proof above, we can classify elements whose scl vanish and list the sources of elements with scl less than 1/48.

Theorem 8.28. *For any 3-manifold M, if a null-homologous element $g \in \pi_1(M)$ represented by a loop γ has $\mathrm{scl}_M(g) < 1/48$, then one of the following cases occurs:*
(1) there is some $n \in \mathbb{Z}_+$ and $h \in \pi_1(M)$ such that $g^n h g^n h^{-1} = id$;
(2) γ up to homotopy is supported in a prime factor of M that admits $\mathbb{S}^3, \mathbb{E}^3, \mathbb{S}^2 \times \mathbb{E}$, Nil, or Sol geometry of finite volume;
(3) γ up to homotopy is supported in a piece with $\mathbb{H}^3$ geometry of finite volume in the (possibly trivial) geometric decomposition of a prime factor of M, such that the geodesic length of γ is less than a universal constant C;

(4) γ up to homotopy is supported in a piece N with $\mathbb{H}^2\times\mathbb{E}$ or $\widetilde{\mathrm{PSL}}_2$ geometry of finite volume in the (possibly trivial) geometric decomposition of a prime factor of M, such that $\mathrm{scl}_{(B,\partial B)}(p(g)) < 1/48$, where $p:\pi_1(N)\to\pi_1(B)$ is the projection induced by the Seifert fibration of N over an orbifold B;

(5) g^2 is represented by a loop in a torus of the JSJ decomposition of a prime factor of M;

(6) g is represented by a loop that is null-homologous in a twisted I-bundle K over the Klein bottle.

Moreover, $\mathrm{scl}_M(g)=0$ if and only if we have cases (1), (2), (6) or the following two special cases of (4) and (5):

(4) $g^nhg^nh^{-1}=z^\ell$ for some $n>0$, $\ell\in\mathbb{Z}$ and $h\in\pi_1(N)$, where z represents a regular fiber, so that either the base space B is nonorientable or N is glued to a twisted I-bundle K over the Klein bottle in the JSJ decomposition such that z is identified with a null-homologous loop in K;*

(5) g^2 is represented by a loop in a JSJ torus that identifies boundary components $\partial_1\subset N_1,\partial_2\subset N_2$ of (possibly the same) pieces N_1,N_2, and $g^2=ab$ so that either $a\in\pi_1(N_1)$ and $b\in\pi_1(N_2)$ represent regular fibers as in case (4*) above or loops as in case (6), or $N_1=N_2$ is Seifert fibered with regular fiber represented by z so that $a=z^m$ and $b=z^{-m}$ for some $m\in\mathbb{Z}$.*

Proof. Suppose $\mathrm{scl}_M(g)<1/48$. If $g^nhg^nh^{-1}=id$ for some $h\in\pi_1(M)$ and $n\in\mathbb{Z}_+$, then $\mathrm{scl}_M(g)=0$. Assume this is not the case in the sequel. Then γ up to homotopy is supported in a prime factor M_0 by Lemma 2.5.

If M_0 has one of the five geometries in case (2), then $\mathrm{scl}_M(g)=\mathrm{scl}_{M_0}(g)=0$ since $\pi_1(M_0)$ is amenable.

If M_0 has hyperbolic geometry, then $\mathrm{scl}_M(g)<1/48$ implies that γ has geodesic length no more than a universal constant C by (the proof of) Theorem 8.10. Moreover, $\mathrm{scl}_M(g)=\mathrm{scl}_{M_0}(g)>0$ since $g\neq id$.

If M_0 has $\mathbb{H}^2\times\mathbb{E}$ or $\widetilde{\mathrm{PSL}}_2$ geometry, then M_0 is Seifert fibered over a closed orbifold B with $\chi_o(B)<0$. The monotonicity of scl implies that $\mathrm{scl}_B(p(g))<1/48$ where $p:\pi_1(M_0)\to\pi_1(B)$ is the projection induced by the Seifert fibration. Moreover, by Theorem 8.27, the case $\mathrm{scl}_M(g)=0$ occurs only if we have case (1) or (4*).

Now suppose M_0 has a nontrivial geometric decomposition. By Theorem 8.26 and our assumption, γ up to homotopy is supported in some piece N of the JSJ decomposition of M_0. Suppose γ is homotopic to a loop in a JSJ torus T, then by Theorem 8.20, $\mathrm{scl}_M(g)=0$ only occurs if we have case (4*) with $n=1$ and $h=id$, case (5*), or case (6).

Suppose γ is not homotopic to a loop in a JSJ torus. Then depending on the type of N we have three cases:

(a) If N has hyperbolic geometry in the interior, then by Theorem 8.10, γ has geodesic length less than the universal constant C mentioned above in the closed hyperbolic case, and $\mathrm{scl}_M(g)>0$ by Lemma 8.13 since γ is not boundary parallel.

(b) If N is Seifert fibered over an orbifold B with $\chi_o(B)<0$, then $p:\pi_1(N)\to\pi_1(B)$ preserves scl since B has boundary, and we have $\mathrm{scl}_{(B,\partial B)}(p(g))=\mathrm{scl}_{(N,\partial N)}(g)\le\mathrm{scl}_{M_0}(g)<1/48$ by Lemma 5.2. Moreover, if $\mathrm{scl}_M(g)=0$, then we have $\mathrm{scl}_{(N,\partial N)}(g)=0$, and thus $\mathrm{scl}_N(g^2-z^m)=0$ and $m\in\mathbb{Z}$ by Lemma 8.14, where z represents a regular fiber. Hence either $m\neq 0$ and $\mathrm{scl}_M(z)=\mathrm{scl}_M(g^2)/m=0$, or $m=0$ and $\mathrm{scl}_N(g)=0$. In the latter case, we have $g^nhg^nh^{-1}=z^\ell$ since $\mathrm{scl}_B(p(g))=0$ where $\ell\neq 0$ by assumption and thus $[z]=0\in H_1(N;\mathbb{R})$ and $\mathrm{scl}_N(z)=0$. In all cases, we have $g^nhg^nh^{-1}=z^\ell$ for some $\ell\neq0$ and $\mathrm{scl}_M(z)=0$. Hence applying Theorem 8.20 to z, we note that $\mathrm{scl}_M(g)=0$ only if we have case (4*).

(c) If N is the twisted I-bundle over the Klein bottle, then g^2 is supported in ∂N. Moreover, $\mathrm{scl}_M(g) = 0$ implies $\mathrm{scl}_M(g^2) = 0$, which happens only if we have case (6) or (5*).

□

Note that there are few conjugacy classes with scl strictly between 0 and 1/48, so one may expect many 3-manifolds to have only finitely many such conjugacy classes. For example, Michael Hull suggested the following result in personal communications.

Corollary 8.29. *Let M be a prime 3-manifold with only hyperbolic pieces in its (possibly trivial) geometric decomposition, then $\mathrm{scl}_M(g) > 0$ for all $g \neq id$ and there are only finitely many conjugacy classes g with $\mathrm{scl}_M(g) < 1/48$.*

Proof. Let $g \neq id$. If g does not conjugate into any vertex group, then $\mathrm{scl}_M(g) \geq 1/6$ by Lemma 8.25. If g lies in a hyperbolic piece N and does not conjugate into any peripheral subgroup, then $\mathrm{scl}_M(g) \geq \mathrm{scl}_{(N,\partial N)}(g) > 0$ by Lemma 8.13. If g conjugates into some JSJ torus, then the proof of Theorem 8.20 shows that scl_M restricted to the edge groups can be computed by a degenerate norm $\|\cdot\|$. Having only hyperbolic pieces implies that $\|\cdot\|$ has trivial vanishing locus by Corollary 8.17, and thus $\mathrm{scl}_M(g) > 0$ and there are only finitely many integer points with norm less than 1/48.

Suppose $\mathrm{scl}_M(g) < 1/48$. Then by our assumption, g must fall into cases (3) or (5) in Theorem 8.28. We have discussed the case where g conjugates into an edge group above. As for the other case, there are certainly only finitely many conjugacy classes supported in a hyperbolic piece of M with short geodesic length. □

One should not expect a similar result in general if we allow Seifert fibered pieces, for the norm $\|\cdot\|$ may have nontrivial vanishing locus. For example, let M_1 be a hyperbolic 3-manifold with one cusp so that a loop γ on the boundary has small positive scl. Let M_2 be a Seifert fibered 3-manifold with nonorientable base space and one torus boundary. Glue M_1 and M_2 along their boundary to obtain M so that γ is not identified with the fiber direction of M_2. Then all elements of the form gz^n have the same small positive scl value in M, where g and z represent the image of γ and the fiber direction of M_2 respectively.

9. Appendix

In this appendix, we prove a uniform spectral gap for scl relative to the (possibly empty) boundary ∂B in any 2-dimensional orbifold B with $\chi_o(B) < 0$. The gap is explicit in its nature, and can be taken to be 1/36 if we exclude orbifolds that have exactly three cone points on a sphere. Non-uniform gaps follow easily from Theorem 8.8. Along the way, the method also implies a uniform gap 1/12 for integral chains in free products of cyclic groups, which might be of independent interest. See Subsection 8.2 for basic definitions about orbifolds. We use scl_B and $\mathrm{scl}_{(B,\partial B)}$ to denote $\mathrm{scl}_{\pi_1(B)}$ and scl relative to the peripheral subgroups respectively.

9.1. Spectral gaps for chains in free products of cyclic groups.

In this subsection we will prove a key result for the spectral gap of orbifolds relative to the boundary. As a byproduct, we use it to prove the following spectral gap for chains in free products of cyclic groups. This is a generalization of [Tao16] uses analogous arguments.

Proposition 9.1. *Let G be an free product of cyclic groups and let $c = \sum c_i g_i$ be an integral chain that is nontrivial in $B_1^H(G)$. Then $\mathrm{scl}_G(c) \geq 1/12$. This gap is sharp on the class of free products of cyclic groups.*

To see that this gap is sharp, let $G = (\mathbb{Z}/2) \star (\mathbb{Z}/3)$ be the free product of the cyclic group of order 2 with generator $\mathtt{a}$ and the cyclic group of order 3 with generator $\mathtt{b}$, then $\mathrm{scl}_G(\mathtt{ab}) = 1/12$ [Cal09b, Theorem 2.93].

Suppose $G = Z_1 \star \cdots \star Z_m \star (\mathbb{Z}/o_1) \star \cdots \star (\mathbb{Z}/o_n)$ with $Z_i \cong \mathbb{Z}$. Let $\mathtt{x}_1, \ldots, \mathtt{x}_m$ be the generators of $Z_1 \star \cdots \star Z_m$ and let $\mathtt{y}_1, \ldots, \mathtt{y}_n$ be the generators of $(\mathbb{Z}/o_1) \star \cdots \star (\mathbb{Z}/o_n)$ with orders $o_1, \ldots, o_n$. Consider the alphabet $S = \{\mathtt{x}_1^{\pm 1}, \ldots, \mathtt{x}_m^{\pm 1}, \mathtt{y}_1^{i_1}, \ldots, \mathtt{y}_n^{i_n}\}$ where $0 < i_j < o_j$ for all $j \in \{1, \ldots, n\}$. We say a word w is reduced if no $\mathtt{x}_j$ is adjacent to $\mathtt{x}_j^{-1}$ and no $\mathtt{y}_j^{i_1}$ is adjacent to $\mathtt{y}_j^{i_2}$. Denote the word length of w by $|w|$, and for any $g \in G$ let $|g| = |w|$ for any cyclically reduced representative w.

Every element $g \in G$ can be uniquely represented by a reduced word $\bar{g}$ in the alphabet S. For two elements $g, h \in G$ we define $C_g(h)$ to be the number of times that $\bar{g}$ occurs as a subword of $\bar{h}$. Finally we set $\phi_g(h) := C_g(h) - C_{g^{-1}}(h)$ analogously to the Brooks quasimorphism first defined by Brooks in [Bro81]. Note that $C_g(h) = C_{g^{-1}}(h^{-1})$.

A word w in S is *self-overlapping* if there are reduced words u, v, where v is non-trivial, such that $w = vuv$ as a reduced word. Note that any periodic word w is self-overlapping.

Lemma 9.2. *Suppose the reduced word $\bar{g}$ representing $g \in G$ is cyclically reduced of length at least 2 and not self-overlapping. Then*

(1) $\phi_g \colon G \to \mathbb{Z}$ is a quasimorphism with $D(\phi) \leq 3$ and its homogenization satisfies $D(\bar{\phi}) \leq 6$.
(2) If g is not conjugate to g^{-1} then $\bar{\phi}_g(g) = 1$.
(3) For every cyclically reduced word w with $|w| \leq |g|$ representing $h \in G$, we have that either w is a cyclic conjugate of $\bar{g}$ or $\bar{g}^{-1}$, or $\bar{\phi}_g(h) = 0$.

This lemma is analogous to [Tao16, Lemmas 3.1 and 3.2], which deals with the torsion-free case. In the absence of torsion one may deduce that $D(\phi_g) \leq 2$, yielding $D(\bar{\phi}_g) \leq 4$.

Proof. Observe that if v, w are reduced words and $\mathtt{z} \in S \cup \{\emptyset\}$ such that $v^{-1} \cdot \mathtt{x} \cdot w$ is reduced then $C_g(v^{-1} \cdot \mathtt{x} \cdot w) = C_g(v^{-1}) + C_g(w) + C_g((v')^{-1} \cdot \mathtt{x} \cdot w')$, where v' is the prefix of length $|g| - 1$ of v if $|v| \geq |g|$ and $v' = v$ otherwise, and w' is defined analogously.

For every $h_1, h_2 \in G$ there are words c_1, c_2, c_3 in G and letters $\mathtt{z}_1, \mathtt{z}_2, \mathtt{z}_3 \in S \cup \{\emptyset\}$ such that

$$\begin{aligned} h_1 &= c_1^{-1} \mathtt{z}_1 c_2 \\ h_2 &= c_2^{-1} \mathtt{z}_2 c_3 \\ (h_1 h_2)^{-1} &= c_3^{-1} \mathtt{z}_3 c_1 \end{aligned}$$

To see bullet (1), we compute that

$$\begin{aligned} \phi_g(h_1) &= C_g(c_1^{-1}) + C_g(c_2) + C_g((c_1')^{-1} \mathtt{z}_1 c_2') - C_{g^{-1}}(c_1^{-1}) - C_{g^{-1}}(c_2) - C_{g^{-1}}((c_1')^{-1} \mathtt{z}_1 c_2') \\ \phi_g(h_2) &= C_g(c_2^{-1}) + C_g(c_3) + C_g((c_2')^{-1} \mathtt{z}_2 c_3') - C_{g^{-1}}(c_2^{-1}) - C_{g^{-1}}(c_3) - C_{g^{-1}}((c_2')^{-1} \mathtt{z}_2 c_3') \\ \phi_g((h_1 h_2)^{-1}) &= C_g(c_3^{-1}) + C_g(c_1) + C_g((c_3')^{-1} \mathtt{z}_3 c_1') - C_{g^{-1}}(c_3^{-1}) - C_{g^{-1}}(c_1) - C_{g^{-1}}((c_3')^{-1} \mathtt{z}_3 c_1'). \end{aligned}$$

Using that $C_g(u) = C_{g^{-1}}(u^{-1})$ we see that

$$\begin{aligned} \phi(h_1) + \phi(h_2) - \phi(h_1 h_2) &= \phi(h_1) + \phi(h_2) + \phi((h_1 h_2)^{-1}) \\ &= C_g((c_1')^{-1} \mathtt{z}_1 c_2') - C_{g^{-1}}((c_1')^{-1} \mathtt{z}_1 c_2') \\ &\quad + C_g((c_2')^{-1} \mathtt{z}_2 c_3') - C_{g^{-1}}((c_2')^{-1} \mathtt{z}_2 c_3') \\ &\quad + C_g((c_3')^{-1} \mathtt{z}_3 c_1') - C_{g^{-1}}((c_3')^{-1} \mathtt{z}_3 c_1'). \end{aligned}$$

All of the terms $(c_j')^{-1} \mathtt{z}_j c_{j+1}'$ have word length strictly less than $2|g|$. As $\bar{g}$ is not self-overlapping we conclude that $C_g((c_j')^{-1} \mathtt{z}_j c_{j+1}') \in \{0, 1\}$. Thus

$$|\phi_g(h_1) + \phi_g(h_2) - \phi_g(h_1 h_2)| \leq 3,$$

and hence $D(\bar{\phi}_g) \leq 2D(\phi_g) \leq 6$. This shows bullet (1).

For bullet (2), observe that $C_g(g^n) = n$, as g is not self-overlapping and is cyclically reduced of length at least 2. Suppose that $C_{g^{-1}}(g^n) \neq 0$. Observe that $\bar{g}$ and $\bar{g}^{-1}$ have the same length and thus $\bar{g}^{-1}$ is already a subword of $\bar{g} \cdot \bar{g}$. We may write $\bar{g} \cdot \bar{g} = u_1 g^{-1} u_2$ as a reduced expression. Since $\bar{g} \cdot \bar{g}$ is $|g|$-periodic, we conclude that $g = u_1 u_2$ and $g^{-1} = u_2 u_1$, thus g and g^{-1} are conjugate.

To see bullet (3), assume that w is any cyclically reduced word with $|w| \leq |g|$ representing $h \in G$. If $|w| = |g|$, then by the same argument as above we see that if $C_g(h^n) \neq 0$, then $\bar{g}$ is a cyclic conjugate of w, and similarly if $C_{g^{-1}}(h^n) \neq 0$. If $|w| < |g|$, suppose that $C_g(h^n) \neq 0$ for some integer $n \in \mathbb{N}$. Because $|w| < |g|$ we have that $n \geq 2$. Then we may write $g = w_e w^m w_s$ as a reduced word where w_e is some suffix of w and w_s is some prefix of w and where $m \geq 0$. Note that g is not a proper power since w is not self-overlapping, either w_e or w_s is nontrivial. If $|w_s|+|w_e| \leq |w|$, then $m \geq 1$ and we may write $g = w_e w_s w' w^{m'} w'' w_e w_s$ for an appropriate choice of w', w'' and m'. This contradicts with the fact that g is not self-overlapping. If $|w_s|+|w_e| > |w|$, then the prefix w_s and the suffix w_e of w intersect in some non-trivial word w_0. Thus we may write $w_e = w_0 w_e'$ and $w_s = w_s' w_0$. Hence $g = w_0 w_e' w^m w_s' w_0$, which again contradicts the fact that g is not self-overlapping. ☐

Lemma 9.3. *Every element $g \in G$ that is not a proper power is conjugate to an element $g' \in G$ such that $\bar{g'}$ is cyclically reduced and not self-overlapping.*

Proof. We follow the strategy of [Tao16, Lemmas 3.1 and 3.2]. Pick an arbitrary total order $\prec$ on S and extend this to a lexicographic order on reduced words and thus on G. Let g' be a conjugate of g such that $\bar{g'}$ is cyclically reduced and such that it is minimal among all conjugates of g with this property. We claim that $\bar{g'}$ not self-overlapping.

If not, then we may write $\bar{g'} = uvu$ as a reduced expression, where u is non-trivial. Then $|uv| = |vu|$ and $uv \neq vu$ since g is not a proper power. If $uv \prec vu$ then uuv is a cyclic conjugate with $uuv \prec uvu$, which contradicts our choice of $\bar{g'}$. Similarly if $vu \prec uv$ then vuu is a cyclic conjugate with $vuu \prec uvu$, which also contradicts our choice of $\bar{g'}$. ☐

We may now prove Proposition 9.1.

Proof of Proposition 9.1. Let $c = \sum_i c_i g_i$ be an integral chain where all c_i are non-zero integers. Up to replacing c by an equivalent integral chain using the defining relation of $B_1^H(G)$ (see Subsection 2.1), we may assume that every $\bar{g}_i$ is cyclically reduced, not proper powers, and none of the g_i are conjugate to each other or their inverses.

Without loss of generality, assume that $|g_1|$ is maximal among all $|g_i|$. We also assume $|g_1| \geq 2$, since otherwise being null-homologous implies that c must be the zero chain. By Lemma 9.3 there is a cyclic conjugate g' of g_1 such that g' is cyclically reduced and not self-overlapping.

Set $\phi = \bar{\phi}_{g'}$. Since g' is conjugate to g_1 and $\bar{\phi}$ is homogeneous, we have $\phi(g_1) = 1$. For every $i > 1$ we have $|g_i| \leq |g_1|$ and $\phi(g_i) = 0$ by Lemma 9.2. By Bavard's duality theorem and the fact that $D(\phi) \leq 6$, we deduce that

$$\mathrm{scl}_G(c) \geq \frac{|\phi(c)|}{2D(\phi)} = \frac{|\sum_i c_i \phi(g_i)|}{2D(\phi)} = \frac{|c_1|}{2D(\phi)} \geq \frac{1}{12}.$$

☐

9.2. **Spectral gap of orbifolds groups relative to the boundary.** In this subsection we will use the quasimorphisms constructed in the previous subsection to give bounds on the stable commutator length of elements in orbifold groups relative to the boundary.

Theorem 9.4. *Let B be an orbifold with boundary. Then for any $g \in \pi_1(B)$, we have either $\mathrm{scl}_{(B,\partial B)}(g) \geq 1/24$ or $\mathrm{scl}_{(B,\partial B)}(g) = 0$. Moreover, $\mathrm{scl}_{(B,\partial B)}(g) = 0$ if and only if*

(1) g has finite order,
(2) g is conjugate to g^{-1}, or

(3) g is represented by a boundary loop.

First consider the orientable case. Let B be a 2-dimensional orientable orbifold of genus k with $m+1$ boundary components and n cone points of orders $o_1,\dots,o_n \in \mathbb{N}$, where $k,m \ge 0$. Then the orbifold fundamental group $G=\pi_1(B)$ is $Z_1 \star \cdots \star Z_{2k} \star Z_1^b \cdots \star Z_m^b \star (\mathbb{Z}/o_1) \star \cdots \star (\mathbb{Z}/o_n)$ where all Z's are infinite cyclic groups. Let $\mathtt{x}_1,\dots,\mathtt{x}_{2k}$ be the natural generating set of $Z_1 \star \cdots \star Z_{2k}$, let $\mathtt{b}_1,\dots,\mathtt{b}_m$ be the natural generating set of $Z_1^b \cdots \star Z_m^b$, and let $\mathtt{y}_1,\dots,\mathtt{y}_n$ be the natural generating set of $(\mathbb{Z}/o_1) \star \cdots \star (\mathbb{Z}/o_n)$. With an appropriate choice, the $m+1$ boundary components with the induced orientation are represented by chains

$$-\mathtt{b}_1, -\mathtt{b}_2, \cdots, -\mathtt{b}_m, [\mathtt{x}_1,\mathtt{x}_2]\cdots[\mathtt{x}_{2k-1},\mathtt{x}_{2k}]\mathtt{b}_1\cdots\mathtt{b}_m\mathtt{y}_1\cdots\mathtt{y}_n.$$

Then the boundary chain ∂B is their sum

$$\partial B := [\mathtt{x}_1,\mathtt{x}_2]\cdots[\mathtt{x}_{2k-1},\mathtt{x}_{2k}]\mathtt{b}_1\cdots\mathtt{b}_m\mathtt{y}_1\cdots\mathtt{y}_n - (\mathtt{b}_1+\cdots+\mathtt{b}_m).$$

Proof of Theorem 9.4 for orientable orbifolds. It is obvious that $\mathrm{scl}_{(B,\partial B)}(g)=0$ in cases (1), (2) and (3). Assuming these are not the case, we now show $\mathrm{scl}_{(B,\partial B)}(g) \ge 1/24$ using the quasimorphisms developed in the previous subsection.

Identify $G=\pi_1(B)$ with the free product of cyclic groups above. Note that the $m+1$ boundary chains span an m-dimensional subspace $V_{\partial B}$ in $H_1(\pi_1(B);\mathbb{R})$, and their only linear combinations that are null-homologous must be of the form $t\cdot\partial B$ for some $t\in\mathbb{R}$.

We assume that $[g]$ lies in $V_{\partial B}$ since otherwise $\mathrm{scl}_{(B,\partial B)}(g)=\infty$. Then there is a unique chain $c=\sum c_i\mathtt{b}_i$ for some $c_i\in\mathbb{R}$ such that $[g]=[c]$, and $\mathrm{scl}_{(B,\partial B)}(g)=\inf_{t\in\mathbb{R}}\mathrm{scl}_B(g-c+t\partial B)$. Note that the assumptions imply that g does not conjugate into any free factor.

Write $g=h^q$ with $q\in\mathbb{Z}_+$ so that h is not a proper power. Let h' be the conjugate of h as in Lemma 9.3. Note that h and h' are not conjugate since g and g^{-1} are not, and that h' has length at least 2 since g does not conjugate into any free factor. Thus $\bar\phi_{h'}(g)=q\ge 1$ and $\bar\phi_{h'}(\mathtt{b}_i)=0$ for all i by Lemma 9.2, where $\bar\phi_{h'}$ is the Brooks quasimorphism constructed in the previous subsection.

Let b be the only word in the chain ∂B of long length, i.e.

$$b := [\mathtt{x}_1,\mathtt{x}_2]\cdots[\mathtt{x}_{2k-1},\mathtt{x}_{2k}]\mathtt{b}_1\cdots\mathtt{b}_m\mathtt{y}_1\cdots\mathtt{y}_n.$$

If $\bar\phi_{h'}(b)=0$ then we compute by Bavard's Duality that

$$\mathrm{scl}(g-c+t\cdot\partial B) \ge \frac{|\bar\phi_{h'}(g-c+t\cdot\partial B)|}{2D(\bar\phi_{h'})} \ge \frac{|\bar\phi_{h'}(g)|}{12} \ge \frac{1}{12}.$$

We are left with the case where $\bar\phi_{h'}(b)\ne 0$. By Lemma 9.2 and the assumption that g is not boundary parallel, this implies $|b|>|h'|\ge 2$ and moreover either h' or its inverse is cyclically a proper subword of b.

From the word we see that this subword is unique, and suppose that h' (instead of h'^{-1}) is this subword, which implies $\bar\phi_{h'}(\partial B)=1$. Observe also that b is not self-overlapping and has length at least 2, thus $\bar\phi_b$ is a quasimorphism of defect at most 6 and $\bar\phi_b(h')=0=\bar\phi_b(\mathtt{b}_i)$ by Lemma 9.2, and hence $\bar\phi_b(g)=0=\bar\phi_b(c)$. Set $\bar\phi := \bar\phi_{h'}-\bar\phi_{b'}$. We may crudely estimate $D(\bar\phi)\le D(\bar\phi_{h'})+D(\bar\phi_b)\le 12$ and compute that $\bar\phi(\partial B)=0$ and $\bar\phi(g)=\bar\phi_{h'}(g)\ge 1$. Thus

$$\mathrm{scl}(g-c+t\partial B) \ge \frac{\bar\phi(g-c+t\partial B)}{2D(\bar\phi)} \ge 1/24,$$

which finishes the proof of the orientable case. □

The nonorientable case is similar. In this case, if B has (nonorientable) genus k and $m+1$ boundary components with cone points of order $o_1,\dots,o_n$, where $k\ge 1, m\ge 0$, then $\pi_1(B)$ is $Z_1\star\cdots\star Z_k\star Z_1^b\cdots\star Z_m^b\star(\mathbb{Z}/o_1)\star\cdots\star(\mathbb{Z}/o_n)$ where all Z's are infinite cyclic groups. Let

$\mathtt{x}_1, \dots, \mathtt{x}_k$ be the natural generating set of $Z_1 \star \cdots \star Z_k$, let $\mathtt{b}_1, \dots, \mathtt{b}_m$ be the natural generating set of $Z_1^b \cdots \star Z_m^b$, and let $\mathtt{y}_1, \dots, \mathtt{y}_n$ be the natural generating set of $(\mathbb{Z}/o_1) \star \cdots \star (\mathbb{Z}/o_n)$. With an appropriate choice, the $m+1$ boundary components are represented by chains

$$-\mathtt{b}_1, -\mathtt{b}_2, \cdots, -\mathtt{b}_m, \mathtt{x}_1^2 \cdots \mathtt{x}_k^2 \mathtt{b}_1 \cdots \mathtt{b}_m \mathtt{y}_1 \cdots \mathtt{y}_n.$$

Proof of Theorem 9.4 for nonorientable orbifolds. The proof is similar to the orientable case. It again suffices to show that $\mathrm{scl}_{(B,\partial B)}(g) \geq 1/24$ assuming that we are not in cases (1), (2) or (3).

Identify $G = \pi_1(B)$ with the free product of cyclic groups above. Now the $m+1$ boundary chains are linearly independent in $H_1(\pi_1(B); \mathbb{R})$ and span an $(m+1)$-dimensional subspace $V_{\partial B}$. We assume that $[g]$ lies in $V_{\partial B}$. Then there is a unique chain $c = c_0 b + \sum c_i \mathtt{b}_i$ for some $c_i \in \frac{1}{2}\mathbb{Z}$ such that $[g] = [c]$, and $\mathrm{scl}_{(B,\partial B)}(g) = \mathrm{scl}_B(g - c)$, where

$$b := \mathtt{x}_1^2 \cdots \mathtt{x}_k^2 \mathtt{b}_1 \cdots \mathtt{b}_m \mathtt{y}_1 \cdots \mathtt{y}_n.$$

Note that the assumptions imply that g does not conjugate into any free factor unless $k = 1$ and g is conjugate to $\mathtt{x}_1^\ell$ for some $\ell \neq 0 \in \mathbb{Z}$. In this exceptional case, $c = \frac{\ell}{2}(b - \sum \mathtt{b}_i)$, and $|\bar{\phi}_b(g - c)| = |\ell|/2$, where $\bar{\phi}_b$ is the Brooks quasimorphism of defect at most 6 by Lemma 9.2 since b is not self-overlapping and has length at least 2. Hence Bavard's duality implies that $\mathrm{scl}_B(g - c) \geq |\ell|/24 \geq 1/24$ in this case.

In the sequel, assume that g does not conjugate into any free factor. Write $g = h^q$ with $q \in \mathbb{Z}_+$ so that h is not a proper power. Let h' be the conjugate of h as in Lemma 9.3. Note that h and h' are not conjugate since g and g^{-1} are not, and that h' has length at least 2 since g does not conjugate into any free factor. Thus $\bar{\phi}_{h'}(g) = q \geq 1$ and $\bar{\phi}_{h'}(\mathtt{b}_i) = 0$ for all i by Lemma 9.2, where $\bar{\phi}_{h'}$ has defect at most 6 by Lemma 9.2.

If $\bar{\phi}_{h'}(b) = 0$ then we compute by Bavard's Duality that

$$\mathrm{scl}(g - c) \geq \frac{|\bar{\phi}_{h'}(g - c)|}{2D(\bar{\phi}_{h'})} \geq \frac{|\bar{\phi}_{h'}(g)|}{12} \geq \frac{1}{12}.$$

We are left with the case where $\bar{\phi}_{h'}(b) \neq 0$. By Lemma 9.2 and the assumption that g is not boundary parallel, this implies $|b| > |h'| \geq 2$ and moreover either h' or its inverse is cyclically a proper subword of b.

From the word we see that this subword is unique since $|h'| > 1$, and suppose that h' (instead of h'^{-1}) is this subword, which implies $\bar{\phi}_{h'}(b) = 1$. Hence $\bar{\phi}_{h'}(g - c) = q - c_0 \in \frac{1}{2}\mathbb{Z}$ and $\mathrm{scl}_B(g - c) \geq |q - c_0|/12 \geq 1/24$ by Bavard's duality unless $c_0 = q$. For this exceptional case, observe that $\bar{\phi}_b$ has defect at most 6 and $\bar{\phi}_b(h') = 0 = \bar{\phi}_b(\mathtt{b}_i)$ by Lemma 9.2, and hence $\bar{\phi}_b(g) = 0, \bar{\phi}_b(c) = c_0$. Set $\bar{\phi} := \bar{\phi}_{h'} - \bar{\phi}_b$, which has defect $D(\bar{\phi}) \leq D(\bar{\phi}_{h'}) + D(\bar{\phi}_{b'}) \leq 12$. Then $\bar{\phi}(c) = c_0 - c_0 = 0$ and $\bar{\phi}(g) = q\bar{\phi}_{h'}(h') - q\bar{\phi}_b(h') = q \geq 1$. Thus

$$\mathrm{scl}(g - c) \geq \frac{\bar{\phi}(g - c)}{2D(\bar{\phi})} \geq 1/24,$$

which completes the proof. □

9.3. **Uniform relative spectral gap of 2-orbifolds.** In this subsection we further deal with closed orbifolds to obtain:

Theorem 9.5. *There is a uniform constant $C > 0$, such that for any compact 2-dimensional orbifold B with $\chi_o(B) < 0$, we have either $\mathrm{scl}_{(B,\partial B)}(g) \geq C$ or $\mathrm{scl}_{(B,\partial B)}(g) = 0$ for all $g \in \pi_1(B)$. Moreover, if B is not the 2-sphere with three cone points, we can take C to be $1/36$.*

Proof. By Theorem 9.4, it suffices to consider the case where B is closed. The case where B is the 2-sphere with three cone points needs more attention and the result is proved below in Theorem 9.7. In the sequel, we assume that this is not the case.

Suppose B is orientable with genus k and n cone points of orders $o_1, \cdots, o_n$. Then $\chi_o(B) = 2 - 2k - \sum(1 - 1/o_i)$ and

$$\pi_1(B) = \langle \mathtt{x}_1, \ldots, \mathtt{x}_{2k}, \mathtt{y}_1, \ldots \mathtt{y}_n \mid \mathtt{y}_i^{o_i} = 1, [\mathtt{x}_1, \mathtt{x}_2] \ldots [\mathtt{x}_{2k-1}, \mathtt{x}_{2k}]\mathtt{y}_1 \ldots \mathtt{y}_n = 1\rangle.$$

If $k \geq 1$, then we rewrite the last relation as $\mathtt{x}_2^{-1}[\mathtt{x}_3, \mathtt{x}_4] \ldots [\mathtt{x}_{2k-1}, \mathtt{x}_{2k}]\mathtt{y}_1 \ldots \mathtt{y}_n = \mathtt{x}_1\mathtt{x}_2^{-1}\mathtt{x}_1^{-1}$, and view $\pi_1(B)$ as the HNN extension over $\mathbb{Z}$ of the free product H generated by $\mathtt{x}_2, \ldots, \mathtt{x}_{2k}, \mathtt{y}_1, \ldots, \mathtt{y}_n$. Since $\chi_o(B) < 0$, by looking at the projection to either $\langle \mathtt{x}_3, \mathtt{x}_4\rangle$ or $\langle \mathtt{y}_1\rangle$, the cyclic subgroups generated by $\mathtt{x}_2^{-1}$ and $\mathtt{x}_2^{-1}[\mathtt{x}_3, \mathtt{x}_4] \ldots [\mathtt{x}_{2k-1}, \mathtt{x}_{2k}]\mathtt{y}_1 \ldots \mathtt{y}_n$ respectively have no conjugates intersecting non-trivially. Thus the $\pi_1(B)$ action on the Bass–Serre tree associated to this HNN extension is 1-acylindrical, and we have a gap $1/24$ for hyperbolic elements by Theorem 8.9.

If $k = 0$, then we have $n \geq 4$ since $\chi_o(B) < 0$ and B is not the 2-sphere with three cone points. Then $\pi_1(B)$ can be viewed as an amalgam over $\mathbb{Z}$ of the free products generated by $\mathtt{y}_1, \mathtt{y}_2$ and $\mathtt{y}_3, \ldots, \mathtt{y}_n$ respectively, where the generator of $\mathbb{Z}$ is sent to $(\mathtt{y}_1\mathtt{y}_2)^{-1}$ and $\mathtt{y}_3 \cdots \mathtt{y}_n$ respectively. Note that $h(\mathtt{y}_1\mathtt{y}_2)^p h^{-1} = (\mathtt{y}_1\mathtt{y}_2)^q$ with $p, q \neq 0$ only when $p = q$ and h is a power of $\mathtt{y}_1\mathtt{y}_2$ unless $o_1 = o_2 = 2$. In the exceptional case, either $n > 4$ or o_3, o_4 are not both equal to 2 since $\chi_o(B) < 0$, either of which implies that $h(\mathtt{y}_3 \cdots \mathtt{y}_n)^p h^{-1} = (\mathtt{y}_3 \cdots \mathtt{y}_n)^q$ with $p, q \neq 0$ only when $p = q$ and h is a power of $\mathtt{y}_3 \cdots \mathtt{y}_n$. It follows that the $\pi_1(B)$ action on the Bass–Serre tree is 2-acylindrical, and we have a gap $1/36$ for hyperbolic elements by Theorem 8.9.

Suppose B is nonorientable with genus k and n cone points of orders $o_1, \cdots, o_n$, where $k \geq 1$. Then $\chi_o(B) = 2 - k - \sum(1 - 1/o_i)$ and

$$\pi_1(B) = \langle \mathtt{x}_1, \ldots, \mathtt{x}_k, \mathtt{y}_1, \ldots \mathtt{y}_n \mid \mathtt{y}_i^{o_i} = 1, \mathtt{x}_2^2 \ldots \mathtt{x}_k^2\mathtt{y}_1 \ldots \mathtt{y}_n = \mathtt{x}_1^{-2}\rangle.$$

The last relation naturally splits $\pi_1(B)$ as an amalgam over $\mathbb{Z}$ of $\langle \mathtt{x}_1\rangle$ and the free product of $\mathtt{x}_2, \ldots, \mathtt{x}_k, \mathtt{y}_1, \ldots, \mathtt{y}_n$. Since $\chi_o(B) < 0$, we know $h(\mathtt{x}_2^2 \ldots \mathtt{x}_k^2\mathtt{y}_1 \ldots \mathtt{y}_n)^p h^{-1} = (\mathtt{x}_2^2 \ldots \mathtt{x}_k^2\mathtt{y}_1 \ldots \mathtt{y}_n)^q$ for $p, q \neq 0$ only when $p = q$ and h is a power of $\mathtt{x}_2^2 \ldots \mathtt{x}_k^2\mathtt{y}_1 \ldots \mathtt{y}_n$. Thus the $\pi_1(B)$ action on the Bass–Serre tree is 2-acylindrical, and we have a gap $1/36$ for hyperbolic elements by Theorem 8.9.

In any of the cases above, if g is an elliptic element in the splitting, we have $\mathrm{scl}_B(g) \geq \mathrm{scl}_{(B', \partial B')}(g)$ by Lemma 5.2, where B' is the sub-orbifold supporting g. Theorem 9.4 implies that $\mathrm{scl}_{(B', \partial B')}(g) \geq 1/24$ unless one of the three exceptional cases occurs. For the exceptional cases (1) and (2), we have $\mathrm{scl}_B(g) = 0$. The remaining case (3) implies that g lies in the edge group $\mathbb{Z} = \langle z\rangle$.

(1) In the HNN extension for B orientable with genus $k \geq 1$ above, nontrivial elements in the edge group has nontrivial homology and thus has $\mathrm{scl}_B(g) = \infty$.
(2) In the amalgam for B orientable with genus $k = 0$ above, let G_1, G_2 be the vertex groups. By Theorem 6.8, we have $\mathrm{scl}_B(z) = \min(\mathrm{scl}_{G_1}(\mathtt{y}_1\mathtt{y}_2), \mathrm{scl}_{G_2}(\mathtt{y}_3 \cdots \mathtt{y}_n))$, which is either 0 or at least $1/12$ by Lemma 2.5 since G_1 and G_2 are free products of cyclic groups.
(3) In the amalgam for B nonorientable above, nontrivial elements in the edge group are nontrivial in the homology and $\mathrm{scl}_B(g) = \infty$ unless $k = 1$, in which case Theorem 6.8 implies $\mathrm{scl}_B(z) = \mathrm{scl}_H(\mathtt{y}_1 \ldots \mathtt{y}_n) \geq 1/12$ by Lemma 2.5 where H is the free product of cyclic groups generated by $\mathtt{y}_i$.

□

In the exceptional case where B is the 2-sphere with three cone points of order p, q, r, we have $1/p + 1/q + 1/r < 1$ when $\chi_o(B) < 0$. In this case the orbifold fundamental group is called the von Dyck group, which is the index two subgroup of the triangle group with parameters p, q, r. The tiling of the hyperbolic plane by hyperbolic triangles of angles $\pi/p, \pi/q, \pi/r$ gives a faithful representation of the triangle group and thus of G.

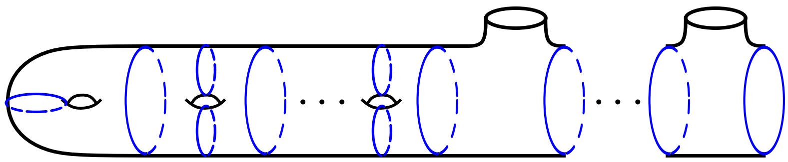

FIGURE 12. A pants of decomposition of S

Lemma 9.6. *There is a uniform constant $\delta := \frac{1}{2}\cosh^{-1}(\cos(2\pi/7)+1/2)$ such that any hyperbolic element in a von Dyck group G has translation length at least δ.*

Proof. For any hyperbolic element $\gamma \in G$, its square γ^2 is hyperbolic in the corresponding triangle group. Thus by [Nak89, Proposition 3.1], the trace of γ^2 as an element of $\mathrm{PSL}_2\mathbb{R}$ satisfies

$$|\mathrm{tr}(\gamma^2)| \geq 2\cos(2\pi/7)+1 = 2\cosh(2\delta).$$

Hence the translation length of γ^2 is at least 2δ, and our conclusion follows. □

Theorem 9.7. *Let G be a von Dyck group. There is a uniform constant $C > 0$ independent of G such that either $\mathrm{scl}_G(\gamma) \geq C$ or $\mathrm{scl}_G(\gamma) = 0$ for any $\gamma \in G$, and the latter case occurs if and only if γ^n is conjugate to γ^{-n} for some n.*

Proof. The proof is simply a modification of the proof of [Cal08, Theorem C] written in [Cal09b, Chapter 3].

Let H be a torsion-free finite index subgroup of G, which we think of as the fundamental group of a hyperbolic closed surface Σ, where the hyperbolic structure is induced from the embedding of G in $\mathrm{PSL}_2\mathbb{R}$.

We estimate scl of any nontrivial $\gamma \in G$. Suppose $\gamma^k = [a_1, b_1]\cdots[a_g, b_g]$. This gives a homomorphism $h : \pi_1(S_g) \cong F_{2g} \to G$, where S_g is a surface of genus g with one boundary. Let S be the finite cover of S_g corresponding to the finite index subgroup $h^{-1}(H)$. In this way, we think of S as a surface mapping (via a map f) into Σ such that each boundary wraps around a loop that is conjugate to some γ^i in G (but not necessarily in H), where $i > 0$.

Now take a pants decomposition of S as in Figure 12. Since all nontrivial elements in H are hyperbolic, after possibly compressing S, we assume each boundary curve of a pair of pants to be hyperbolic. By an argument similar to [Cal09b, Lemma 3.7], we can modify the map f on S by Dehn twist,s after possibly further compressing S, so that f restricted to each pair of pants on S does not factor through a circle.

It is possible that S contains disk or annuli, in which case, some powers of γ are conjugate to its inverse. Assuming this is not case, then a theorem of Thurston (see for example [Cal09b, Lemma 3.6]) shows that S has a pleated surface representative (see the proof of Theorem 8.10 for a definition) and $-\chi(S)/n = -\chi^-(S)/n \leq -\chi(S_g)/k$.

Choose ϵ small compared to the 2-dimensional Margulis constant and $\mathrm{length}(\gamma)$ and take the 2ϵ thin-thick decomposition on the double of S. The exact same argument in the proof of Theorem 8.10 shows that there is some segment σ of ∂S in the thin part satisfies

$$\mathrm{length}(\sigma) \geq \frac{n\cdot \mathrm{length}(\gamma) - \frac{-4\pi\chi(S)}{\epsilon}}{-6\chi(S)} = \frac{n\cdot\mathrm{length}(\gamma)}{-6\chi(S)} - \frac{2\pi}{3\epsilon}.$$

On the other hand, $\mathrm{length}(\sigma)$ cannot be much longer than $\mathrm{length}(\gamma)$. More precisely, choosing ϵ small in the beginning so that 4ϵ is less than the 2-dimensional Margulis constant, it is shown in the original proof of [Cal09b, Theorem 3.9] that

$$\mathrm{length}(\sigma) \leq 2\cdot\mathrm{length}(\gamma) + 4\epsilon$$

unless there are conjugates a, a' of γ in G and $p \in \mathbb{H}^2$ such that $d(p, aa'(p)) \leq 4\epsilon$ and $d(p, a'a(p)) \leq 4\epsilon$.

By Margulis' lemma, aa' and $a'a$ lie in an almost-nilpotent subgroup N of G. Note that any nilpotent group has a nontrivial center. By analyzing centralizers of elements in $\mathrm{PSL}_2\mathbb{R}$, we observe that discrete nilpotent subgroups are cyclic. Since each hyperbolic element has a unique m-th root, we see that if aa' is hyperbolic then aa' and $a'a$ are commuting conjugate hyperbolic elements. In this case the original proof of [Cal09b, Theorem 3.9] shows that $b\gamma^2 b^{-1} = \gamma^{-2}$ for some b.

If aa' is elliptic fixing some $z \in \mathbb{H}^2$, then $a'a$ is also elliptic with the same angle of rotation fixing $a'(z)$, which is distinct from z since a' is hyperbolic. When 4ϵ is small compared to $\mathrm{length}(\gamma)$, since $d(z, a'(z)) \geq \mathrm{length}(\gamma)$, the only way to have a common p with both $d(p, aa'(p)) \leq 4\epsilon$ and $d(p, a'a(p)) \leq 4\epsilon$ is when aa' rotates by a very small angle at the scope of ϵ. But then one can easily observe that $(aa')(a'a)^{-1} = [a, a'] \in G$ has a fixed point on $\partial\mathbb{H}^2$, which implies either a and a' commute or $[a, a']$ is a hyperbolic element. The former case cannot happen since a, a' have distinct fixed points and rotate by a small angle. In the latter case $[a, a']$ has translation length at a scale no more than ϵ.

Since translation lengths of hyperbolic elements in von Dyck groups have a uniform positive lower bound by Lemma 9.6, one can choose a uniform $\epsilon > 0$ that is small compared to $\mathrm{length}(\gamma)$ for any hyperbolic element $\gamma \in G$, which would exclude the case above that $[a, a']$ is hyperbolic.

Hence in this case we must have

$$\mathrm{length}(\sigma) \leq 2 \cdot \mathrm{length}(\gamma) + 4\epsilon.$$

Combining the estimates, we get

$$\left(\frac{n}{-6\chi^-(S)} - 2 \right) \mathrm{length}(\gamma) \leq 4\epsilon + \frac{2\pi}{3\epsilon}.$$

Since ϵ is uniform and $\mathrm{length}(\gamma)$ has a uniform positive lower bound, this yields a uniform positive lower bound of $-\chi^-(S)/n \leq -\chi^-(S_g)/k$, and thus a uniform lower bound C of $\mathrm{scl}_G(\gamma)$ whenever γ is hyperbolic and has no power conjugates to its inverse. □

References

[ADŠ18] Yago Antolín, Warren Dicks, and Zoran Šunić. Left relativity convex subgroups. In *Topological methods in group theory*, volume 451 of *London Math. Soc. Lecture Note Ser.*, pages 1–18. Cambridge Univ. Press, Cambridge, 2018.

[Bav91] Christophe Bavard. Longueur stable des commutateurs. *Enseign. Math. (2)*, 37(1-2):109–150, 1991.

[BBF16] Mladen Bestvina, Ken Bromberg, and Koji Fujiwara. Stable commutator length on mapping class groups. *Ann. Inst. Fourier (Grenoble)*, 66(3):871–898, 2016.

[BFH16] Michelle Bucher, Roberto Frigerio, and Tobias Hartnick. A note on semi-conjugacy for circle actions. *Enseign. Math.*, 62(3-4):317–360, 2016.

[BH11] Michael Björklund and Tobias Hartnick. Biharmonic functions on groups and limit theorems for quasimorphisms along random walks. *Geom. Topol.*, 15(1):123–143, 2011.

[BM99] M. Burger and N. Monod. Bounded cohomology of lattices in higher rank Lie groups. *J. Eur. Math. Soc. (JEMS)*, 1(2):199–235, 1999.

[BM02] M. Burger and N. Monod. Continuous bounded cohomology and applications to rigidity theory. *Geom. Funct. Anal.*, 12(2):219–280, 2002.

[Bro81] Robert Brooks. Some remarks on bounded cohomology. In *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, volume 97 of *Ann. of Math. Stud.*, pages 53–63. Princeton Univ. Press, Princeton, N.J., 1981.

[Cal08] Danny Calegari. Length and stable length. *Geom. Funct. Anal.*, 18(1):50–76, 2008.

[Cal09a] Danny Calegari. Faces of the scl norm ball. *Geom. Topol.*, 13(3):1313–1336, 2009.

[Cal09b] Danny Calegari. *scl*, volume 20 of *MSJ Memoirs*. Mathematical Society of Japan, Tokyo, 2009.

[CF10] Danny Calegari and Koji Fujiwara. Stable commutator length in word-hyperbolic groups. *Groups Geom. Dyn.*, 4(1):59–90, 2010.

[CFL16] Matt Clay, Max Forester, and Joel Louwsma. Stable commutator length in Baumslag-Solitar groups and quasimorphisms for tree actions. *Trans. Amer. Math. Soc.*, 368(7):4751–4785, 2016.

[Che18a] Lvzhou Chen. Scl in free products. *Algebr. Geom. Topol.*, 18(6):3279–3313, 2018.

[Che18b] Lvzhou Chen. Spectral gap of scl in free products. *Proc. Amer. Math. Soc.*, 146(7):3143–3151, 2018.

[Che19] Lvzhou Chen. Scl in graphs of groups, 2019.

[DH91] Andrew J. Duncan and James Howie. The genus problem for one-relator products of locally indicable groups. *Math. Z.*, 208(2):225–237, 1991.

[DM84] Michael W. Davis and John W. Morgan. Finite group actions on homotopy 3-spheres. 112:181–225, 1984.

[FFT19] Talia Fernós, Max Forester, and Jing Tao. Effective quasimorphisms on right-angled Artin groups. *Ann. Inst. Fourier (Grenoble)*, 69(4):1575–1626, 2019.

[FM98] Benson Farb and Howard Masur. Superrigidity and mapping class groups. *Topology*, 37(6):1169–1176, 1998.

[Fri17] Roberto Frigerio. *Bounded cohomology of discrete groups*, volume 227 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2017.

[FSTar] Max Forester, Ignat Soroko, and Jing Tao. Genus bounds in right-angled artin groups. *Publ. Mat.*, To appear.

[Ghy87] Étienne Ghys. Groupes d'homéomorphismes du cercle et cohomologie bornée. In *The Lefschetz centennial conference, Part III (Mexico City, 1984)*, volume 58 of *Contemp. Math.*, pages 81–106. Amer. Math. Soc., Providence, RI, 1987.

[Gre90] Elisabeth Ruth Green. *Graph products of groups*. PhD thesis, University of Leeds, 1990.

[Heu19] Nicolaus Heuer. Gaps in SCL for amalgamated free products and RAAGs. *Geom. Funct. Anal.*, 29(1):198–237, 2019.

[Hol75] Richard B. Holmes. *Geometric functional analysis and its applications*. Springer-Verlag, New York-Heidelberg, 1975. Graduate Texts in Mathematics, No. 24.

[Hub13] T. Huber. Rotation quasimorphisms for surfaces. *PhD Thesis, ETH Zurich*, 2013.

[IK18] Sergei V. Ivanov and Anton A. Klyachko. Quasiperiodic and mixed commutator factorizations in free products of groups. *Bull. Lond. Math. Soc.*, 50(5):832–844, 2018.

[IMT19] Tetsuya Ito, Kimihiko Motegi, and Masakazu Teragaito. Generalized torsion and decomposition of 3–manifolds. *Proc. Amer. Math. Soc.*, 147(11):4999–5008, 2019.

[Joh79] Klaus Johannson. *Homotopy equivalences of* 3*-manifolds with boundaries*, volume 761 of *Lecture Notes in Mathematics*. Springer, Berlin, 1979.

[JS79] William Jaco and Peter B. Shalen. Seifert fibered spaces in 3-manifolds. In *Geometric topology (Proc. Georgia Topology Conf., Athens, Ga., 1977)*, pages 91–99. Academic Press, New York-London, 1979.

[KM96] Vadim A. Kaimanovich and Howard Masur. The Poisson boundary of the mapping class group. *Invent. Math.*, 125(2):221–264, 1996.

[Nak89] Toshihiro Nakanishi. The lengths of the closed geodesics on a Riemann surface with self-intersections. *Tohoku Math. J. (2)*, 41(4):527–541, 1989.

[Per02] Grisha Perelman. The entropy formula for the ricci flow and its geometric applications, 2002.

[Per03a] Grisha Perelman. Finite extinction time for the solutions to the ricci flow on certain three-manifolds, 2003.

[Per03b] Grisha Perelman. Ricci flow with surgery on three-manifolds, 2003.

[Sco83] Peter Scott. The geometries of 3-manifolds. *Bull. London Math. Soc.*, 15(5):401–487, 1983.

[Ser80] Jean-Pierre Serre. *Trees*. Springer-Verlag, Berlin-New York, 1980. Translated from the French by John Stillwell.

[Tao16] Jing Tao. Effective quasimorphisms on free chains. *arXiv e-prints*, page arXiv:1605.03682, May 2016.

[Thu78] William Thurston. The geometry and topology of 3-manifolds. *Lecture note*, 1978.

[Thu82] William P. Thurston. Three-dimensional manifolds, Kleinian groups and hyperbolic geometry. *Bull. Amer. Math. Soc. (N.S.)*, 6(3):357–381, 1982.

[Thu97] William P. Thurston. *Three-dimensional geometry and topology. Vol. 1*, volume 35 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1997. Edited by Silvio Levy.

[WZ10] Henry Wilton and Pavel Zalesskii. Profinite properties of graph manifolds. *Geom. Dedicata*, 147:29–45, 2010.

Department of Mathematics, University of Chicago, Chicago, Illinois, USA
E-mail address, L. Chen: lzchen@math.uchicago.edu

Department of Pure Mathematics and Mathematical Statistics, Centre for Mathematical Sciences, University of Cambridge
E-mail address, N. Heuer: nh441@cam.ac.uk