Scl in graphs of groups

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Let G be a group acting on a tree with cyclic edge and vertex stabilizers. Then stable commutator length (scl) is rational in G. Furthermore, scl varies predictably and converges to rational limits in so-called "surgery" families. This is a homological analog of the phenomenon of geometric convergence in hyperbolic Dehn surgery.

Contents

1 Introduction

If X is a space, there is a (pseudo-)norm on $H_2(X)$ whose value on an integral class α is the infimum of $-\chi(S)/2n$ over all $n > 0$ and all aspherical surfaces S representing $n\alpha$. This is called the Gromov–Thurston norm. If X is a 3-manifold the unit ball is a rational polyhedron [31].

There is a relative Gromov norm called *stable commutator length* (hereafter scl), which measures the minimal complexity of a surface with prescribed boundary. More precisely, scl of a null-homologous loop γ is the infimum of $-\chi(S)/2n$ over all $n > 0$ and all surfaces S that contains no sphere or disk components and whose boundary represents $n\gamma$. It only depends on π_1, and can therefore be thought of as a function on groups, where it can be given a purely algebraic definition. Unfortunately exact calculations are very hard and it is not known how to compute scl even in a closed surface group.

Calegari [5] gave an algorithm to compute scl in free groups. The nature of the algorithm shows the unit ball is a rational polyhedron, and Calegari asked if the same might be true for wider classes of groups, eg hyperbolic groups or groups acting nicely on trees.

In this paper, we give a positive answer for groups acting on trees when edge and vertex stabilizers are cyclic. This is a large and interesting class of groups, including eg the Baumslag–Solitar groups (some authors prefer to use t^{-1} as the generator)

$$\mathrm{BS}(M, L) := \left\langle a, t \mid a^M = t a^L t^{-1} \right\rangle.$$

The method of proof is very geometric, and our techniques give us a way of understanding all (relative) maps of surface pairs $(S, \partial S)$ to 2-complexes presenting such groups. In the case of free groups this is analogous to eg the normal forms obtained by Wicks [33] and Culler [21].

There is a tension here. The set of all homotopy classes of maps of all surfaces to a fixed 2-complex has a lot of information but very little structure. More structure arises when we allow more operations: cut-and-paste, compression, and passing to finite covers. The last operation is rather analogous to working over $\mathbb{Q}$ instead of $\mathbb{Z}$, and indeed mirrors the relative complexity of linear and integer programming respectively.

A group acting on a tree is encoded by a graph of groups. Our groups are those whose edge and vertex groups are cyclic. For a fixed graph one obtains families of groups, by varying the edge homomorphisms. This gives rise to a linear structure on this set, and it makes sense to study how scl varies with respect to these parameters.

For certain families—so called surgery families—we are able to show that scl varies *quasirationally*, ie as a ratio of quasipolynomials. Analogous phenomena arise in Ehrhart theory [22] and in parametric families of integer programming problems [11].

1.1 Statement of results

1.1.1 Rationality

In this paper we develop a method to compute scl in many graphs of groups using linear programming. The nature of the algorithm shows that scl is piecewise rational linear on chains (See Subsection 2.1 for definitions).

The main result of the paper is:

Theorem 5.10 (rationality) Let G be a graph of groups with vertex groups $\{G_v\}$ and edge groups $\{G_e\}$ where

  (1) $\mathrm{scl}_{G_v} \equiv 0$; and

(2) the images of the edge groups in each vertex group are *central* and *mutually commensurable*.

Then scl_G is piecewise rational linear, and $\mathrm{scl}_G(c)$ can be computed via linear programming for each rational chain $c \in B_1^H(G)$.

The theorem above applies to many interesting groups. Bullet (1) holds true for amenable groups [7, Theorem 2.47], irreducible lattices in higher rank Lie groups [3, 4] (see also [7, Theorem 5.26]), and certain transformation groups like $\mathrm{Homeo}^+(\mathrm{S}^1)$ [23, Proposition 5.11] and subgroups of $\mathrm{PL}^+(\mathrm{I})$ [8, Theorem A]. In particular, Theorem 5.10 applies to all graphs of groups with vertex and edge groups isomorphic to $\mathbb{Z}$ (also known as generalized Baumslag–Solitar groups). See Example 4.1 for more groups covered by this theorem.

A map of a surface group to a graph of groups is represented geometrically by a map of a surface to a graph of spaces. The surface can be cut into pieces along curves mapping to the edge spaces. If we understand the edge groups we can give conditions under which such pieces may be reassembled. When edge groups are infinite, such gluing conditions depend a priori on an infinite amount of data which we refer to as "winding numbers".

The key to our method is to keep track of a *sufficient* but *finite* amount of information about winding numbers. What makes this approach possible is a method to solve gluing conditions *asymptotically*. This is a geometric covering space technique and depends on residual properties of the fundamental group of the underlying graph. It also relies on an elementary but crucial observation about *stability* of *virtual isomorphisms* of abelian groups (see Subsection 4.1).

1.1.2   Extremal surfaces

Definition 1.1 If $-\chi(S)/2n$ achieves the infimum in the definition of scl, we say S is *extremal*.

Extremal surfaces are π_1-injective. Techniques developed to find them have lead to significant progress on Gromov's question of finding surface subgroups in hyperbolic groups [9, 13, 15, 34]. A necessary condition for an extremal surface to exist is that $\mathrm{scl} \in \mathbb{Q}$. Thus a natural question in view of Theorem 5.10 is whether extremal surfaces exist for rational chains.

The answer is negative in general, but we have an algorithmic criterion (Theorem 6.20) in the special case of Baumslag–Solitar groups. For any *reduced* rational chain c,

the criterion for the existence of extremal surfaces is expressed in terms of branched surfaces canonically built from the result of the linear programming problem computing $\mathrm{scl}_{\mathrm{BS}(M,L)}(c)$. Here a rational chain $c = \sum r_i g_i$ with $r_i \in \mathbb{Q}_{>0}$ is *reduced* if no positive powers of g_i and g_j "almost" cobound an annulus (See Definition 6.6 and Lemma 6.7).

To describe a special case, let $h : \mathrm{BS}(M,L) \to \mathbb{Z}$ be the homomorphism with $h(a) = 0$ and $h(t) = 1$ in terms of the standard presentation. Its kernel $\ker h = \langle a_k, k \in \mathbb{Z} \mid a_k^M = a_{k+1}^L \rangle$ sits inside $\mathrm{BS}(M,L)$ via $a_k \mapsto t^k a t^{-k}$.

Corollary 6.16 (extremal surfaces) Let $M \neq \pm L$. For a reduced rational chain $c = \sum r_i g_i$, an extremal surface exists if each g_i either has $h(g_i) \neq 0$ or is of the form $a_{k_1}^{u_1} \ldots a_{k_n}^{u_n} \in \ker h$ with

$$\sum_j u_j \left(\frac{M}{L}\right)^{k_j} = 0.$$

1.1.3 Comparison theorems

There is a striking relation between scl in $\mathrm{BS}(M,L)$ and in $\mathbb{Z}/M\mathbb{Z} * \mathbb{Z}/L\mathbb{Z}$. A word $w \in \mathrm{BS}(M,L) = \langle a,t \mid a^M = ta^L t^{-1} \rangle$ is *t-alternating* if it can be written as $w = a^{u_1} t a^{v_1} t^{-1} a^{u_2} t a^{v_2} t^{-1} \cdots a^{u_n} t a^{v_n} t^{-1}$, where the generator t alternates between t and t^{-1}.

Corollary 3.12 Let $\mathbb{Z}/M\mathbb{Z} * \mathbb{Z}/L\mathbb{Z} = \langle x,y \mid x^M = y^L = 1 \rangle$. For any *t*-alternating word in $\mathrm{BS}(M,L)$, we have

$$\mathrm{scl}_{\mathrm{BS}(M,L)}(a^{u_1} t a^{v_1} t^{-1} a^{u_2} t a^{v_2} t^{-1} \cdots a^{u_n} t a^{v_n} t^{-1}) = \mathrm{scl}_{\mathbb{Z}/M\mathbb{Z} * \mathbb{Z}/L\mathbb{Z}}(x^{u_1} y^{v_1} \cdots x^{u_n} y^{v_n}).$$

For any group G, the *scl spectrum* is the image of scl_G in $\mathbb{R}$. Corollary 3.12 implies that the spectrum of $\mathrm{BS}(M,L)$ contains the spectrum of $\mathbb{Z}/M\mathbb{Z} * \mathbb{Z}/L\mathbb{Z}$. Proposition 3.11 is the more general statement with $\mathbb{Z}$ replaced by an arbitrary abelian group. This is a special case of the isometric embedding theorems we prove in Section 3.

A different kind of relationship is expressed in the next theorem. The family of Baumslag–Solitar groups $\mathrm{BS}(M,L)$ with the obvious generators converges to the free group F_2 as marked groups [16] when $M,L \to \infty$. This convergence is reflected in the behavior of scl.

Theorem 6.29 (convergence) For any chain $c \in B_1^H(F_2)$, let $\bar{c}$ be its image in $\mathrm{BS}(M,L)$. If $\gcd(M,L) \to \infty$, then $\mathrm{scl}_{\mathrm{BS}(M,L)}(\bar{c})$ converges to $\mathrm{scl}_{F_2}(c)$. Moreover, for M,L fixed, the sequence $\mathrm{scl}_{\mathrm{BS}(dM,dL)}(\bar{c})$ is eventually quasirational in d and converges to $\mathrm{scl}_{F_2}(c)$.

This resembles the geometric convergence witnessed in hyperbolic Dehn surgery. However in Theorem 6.29, it is important for the parameters (M, L) to go to infinity in a specific way. An example (Proposition 6.25) shows that such an assumption is necessary, and scl is not a continuous function on the space of marked groups.

1.2 Comparison with previous results

Here is a list of groups where scl is previously shown to be piecewise rational linear.

(1) Free groups, by Calegari [5].

(2) Free products of cyclic groups, by Walker [32].

(3) Free products of free abelian groups, by Calegari [6].

(4) Free products $*_\lambda G_\lambda$ with $\mathrm{scl}_{G_\lambda} \equiv 0$ for all λ, by the author [17].

(5) Amalgams of free abelian groups, by Susse [28].

(6) t-alternating words in Baumslag–Solitar groups, by Clay–Forester–Louwsma [20].

Theorem 5.10 is a generalization of all the rationality results above. Corollary 3.12 provides an easier way to understand and compute scl of t-alternating words in Baumslag–Solitar groups.

Regarding extremal surfaces, they exist for any rational chains in bullet (1) and (2). In all other results, extremal surfaces do not exist in general. In bullet (6), however, a criterion [20, Theorem 5.7] is provided for t-alternating words that bound extremal surfaces. Our criterion (Theorem 6.20) extends this to general words and rational chains.

A few previous results are similar to the convergence Theorem 6.29, expressing scl in a group G as the limit of scl in a certain family of quotients of G.

(1) [6, Theorem 4.13] deals with $G = A * B \to A' * B$ where A, A', B are free abelian with $\mathrm{rank}(A) = \mathrm{rank}(A') + 1$ and the maps are induced by a linear family of surjective homomorphisms $A \to A'$ and $id : B \to B$;

(2) [28, Corollary 4.12] handles the case $G = F_2 = \langle x, y \rangle \to \mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}$ by adding the relation $x^M = y^L$ with $M, L \to \infty$; and

(3) [17, Theorem 6.4] proves the case $G = F_2 = \langle x, y \rangle \to \mathbb{Z}/M\mathbb{Z} * \mathbb{Z}/L\mathbb{Z}$ by adding the relation $x^M = y^L = 1$ with $M, L \to \infty$.

The convergences in these results are all quasirational in the parameters. Note that how M, L go to infinity matters in Theorem 6.29 but not in the results above. These results together suggest a more general phenomenon to be discovered.

1.3 Organization of the paper

To ease into our discussion, we introduce the notion of *relative stable commutator length* and develop its basic properties in Section 2, where we also review some elements of scl and graphs of groups. In Section 3 we study surfaces in graphs of groups and their (simple) normal forms, from which we obtain two isometric embedding theorems. In Section 4 we define disk-like pieces. These hold a finite amount of information about winding numbers, which turns out to be sufficient to solve the gluing conditions asymptotically. Finally in Section 6, we focus on the case of Baumslag–Solitar groups, investigating when the asymptotic realization can terminate at a finite stage resulting in extremal surfaces. Then we establish explicit formulas, prove the convergence theorem, and give an implementation with low time complexity.

1.4 Acknowledgment

2 Background

2.1 Stable commutator length

We review some basics of scl (stable commutator length) and set up notation. For a group G, let $C_1(G)$ be the 1-chains in G, namely the $\mathbb{R}$-vector space with basis G. Taking group homology is a linear map $h_G : C_1(G) \to H_1(G; \mathbb{R})$ whose kernel contains $H(G)$, the subspace of $C_1(G)$ spanned by elements of forms $g^n - ng$ or $hgh^{-1} - g$ for some $g, h \in G$ and $n \in \mathbb{Z}$. Thus, on the quotient $C_1^H(G) := C_1(G)/H(G)$, we have a well-defined linear map $\bar{h}_G : C_1^H(G) \to H_1(G; \mathbb{R})$. Denote the kernel of $\bar{h}_G$ by $B_1^H(G)$.

Definition 2.1 Let X be a $K(G, 1)$ space. For an integral chain $c = \sum_{i \in I} g_i$, let γ_i be a loop in X representing g_i. An admissible surface for c is a map $f : (S, \partial S) \to (X, \sqcup \gamma_i)$ from a compact oriented surface S, such that the following diagram commutes and

$\partial f_*[\partial S] = n(S)[\sqcup S^1]$.

$$
\begin{array}{ccc}
\partial S & \xrightarrow{\ i\ } & S \\
\partial f \downarrow & & f \downarrow \\
\sqcup S^1 & \xrightarrow{\ \sqcup \gamma_j\ } & X
\end{array}
$$

For a rational chain c, a certain multiple mc with $m \in \mathbb{Z}_+$ is integral. A surface S is admissible for c of degree n if it is admissible for some mc of degree k such that $n = mk$.

Admissible surfaces exist when the rational chain c is null-homologous, ie $c \in B_1^H(G)$. We measure the complexity of a surface S by $-\chi^-(S)$, where $\chi^-(\Sigma) = \min(0, \chi(\Sigma))$ for each connected component Σ, and $\chi^-(S)$ is the sum of $\chi^-(\Sigma)$ over all its components. Equivalently, $\chi^-(S)$ is the Euler characteristic of S neglecting sphere and disk components.

Definition 2.2 The *stable commutator length* of a null-homologous rational chain c, denoted $\mathrm{scl}_G(c)$, is defined as

$$
\mathrm{scl}_G(c) := \inf_S \frac{-\chi^-(S)}{2n(S)},
$$

where the infimum is taken over all admissible surfaces S of degree $n(S) \geq 1$.

This agrees with the algebraic definition of scl as the limit of commutator lengths. See [7, Chapter 2].

Throughout this paper, we only consider admissible surfaces where each boundary component is an orientation preserving covering map of some loop in the chain. This does not affect the computation of scl [7, Proposition 2.13].

It immediately follows by pushing forward admissible surfaces that scl is *monotone*: For any homomorphism $\phi : G \to H$, we have $\mathrm{scl}_G(c) \geq \mathrm{scl}_H(\phi(c))$ for any chain c. The equality holds if ϕ is an isomorphism, or more generally, if ϕ admits a retract. In particular, scl is *invariant under conjugation*.

Scl extends continuously in a unique way to all null-homologous real chains and induces a pseudo-norm on $B_1^H(G)$ [7, Chapter 2]. As a consequence, removing finite order elements from the chain c does not affect $\mathrm{scl}_G(c)$. We make the convention that $\mathrm{scl}_G(c) = +\infty$ if c has non-trivial homology.

We say scl_G is *piecewise rational linear* if it is a piecewise rational linear function on each rational finite-dimensional subspace of $B_1^H(G)$.

Definition 2.3 We say a homomorphism $\phi : G \to H$ is an *isometric embedding*, if ϕ is injective and

$$\mathrm{scl}_G(c) = \mathrm{scl}_H(\phi(c)),$$

for all $c \in B_1^H(G)$.

Example 2.4 Here are some occasions where we know an injection $\phi : G \to H$ is an isometric embedding.

(1) ϕ admits a retract $r : H \to G$, that is $r \circ \phi = id$. This follows from monotonicity.

(2) G is abelian. In this case, we have $B_1^H(G) = 0$.

(3) $\mathrm{scl}_G \equiv 0$.

Isometric embeddings allow us to pull back admissible surfaces at arbitrarily small cost.

Lemma 2.5 Let $\phi : G \to H$ be an isometric embedding, realized by a map $\phi : X_G \to X_H$ between $K(G, 1)$ and $K(H, 1)$ spaces. Suppose $f' : S' \to X_H$ is a surface without sphere components in X_H such that $\partial f' \partial S' = \phi(\underline{\gamma})$ for a collection of loops $\underline{\gamma}$ in X_G whose sum is a null-homologous chain c. Then for any $\epsilon > 0$, there is a surface $f : S \to X_G$ satisfying the following properties:

(1) *S has no sphere components;*

(2) *There is a (disconnected) covering map $\pi : \partial S \to \partial S'$ of a certain degree $n > 0$ such that the following diagram commutes;*

$$
\begin{array}{ccc}
\partial S & \xrightarrow{\ \pi\ } & \partial S' \\
{\scriptstyle \partial f}\downarrow & & {\scriptstyle \partial f'}\downarrow \\
X_G & \xrightarrow{\ \phi\ } & X_H
\end{array}
$$

(3) *We have*

$$\frac{-\chi(S)}{n} \le -\chi(S') + \epsilon.$$

Proof Let D' be the union of disk components in S'. Since ϕ is an isometric embedding and thus π_1-injective, the loops in $\underline{\gamma}$ corresponding to $\partial D'$ bound a collection of disks D in X_G accordingly. Let S_0' be the remaining components of S'. Then $\partial S_0'$ represents a null-homologous chain c_0 equivalent to c in $B_1^H(G)$ and $\chi(S_0') = \chi^-(S_0')$. Since ϕ preserves scl, there is an admissible surface S_0 for c_0 of a certain degree $n > 0$ without disk or sphere components such that

$$\frac{-\chi(S_0)}{n} \le 2\mathrm{scl}_G(c_0) + \epsilon = 2\mathrm{scl}_H(\phi(c_0)) + \epsilon \le -\chi(S_0') + \epsilon.$$

Then the surface $S = S_0 \sqcup nD$ has the desired properties. $\qquad\square$

2.2 Relative stable commutator length

It is natural and convenient for our discussion to introduce *relative stable commutator length*. Several previous work [10, 20, 17, 24] contains similar thoughts without formally formulating this notion.

Definition 2.6 Let $\mathcal{G} = \{G_\lambda\}_{\lambda \in \Lambda}$ be a collection of subgroups of a group G. For a chain $c \in C_1(G)$, its *stable commutator length relative to $\mathcal{G}$* is

$$\operatorname{scl}_{(G,\mathcal{G})}(c) := \inf \left\{ \operatorname{scl}\left(c + \sum c_\lambda \right) : c_\lambda \in C_1(G_\lambda) \right\},$$

where each summation contains only finitely many non-zero c_λ.

Recall that we have a linear map $\bar{h}_G : C_1^H(G) \to H_1(G; \mathbb{R})$ whose kernel is denoted as $B_1^H(G)$ in Subsection 2.1. Let $H_1(\mathcal{G}) \le H_1(G)$ be the image of the inclusion $\oplus_\lambda H_1(G_\lambda) \to H_1(G)$. Denote $\bar{h}_G^{-1} H_1(\mathcal{G})$ by $B_1^H(G, \mathcal{G})$, which contains $B_1^H(G)$ as a subspace.

We summarize some basic properties of relative scl in the following lemma.

Lemma 2.7 *With notation as above, we have:*

(1) *A chain c has finite $\operatorname{scl}_{(G,\mathcal{G})}(c)$ if and only if its homology class $[c]$ lies in $H_1(\mathcal{G})$.*

(2) *$\operatorname{scl}_{(G,\mathcal{G})}$ is a well defined pseudo norm on $B_1^H(G, \mathcal{G})$.*

(3) *$\operatorname{scl}_G(c) \ge \operatorname{scl}_{(G,\mathcal{G})}(c)$.*

(4) *For any homomorphism $\phi : G \to H$, we have*

$$\operatorname{scl}_{(G,\mathcal{G})}(c) \ge \operatorname{scl}_{(H,\phi(\mathcal{G}))}(c).$$

If ϕ is an isomorphism, then equality holds.

(5) *If g conjugates into some G_λ, then $\operatorname{scl}_{(G,\mathcal{G})}(g) = 0$.*

When c is a rational chain, its relative scl can be described in terms of *relative admissible surfaces* as follows.

Definition 2.8 Let X be a topological space with subspaces X_λ such that $\pi_1(X) \cong G$ and under this isomorphism $\pi_1(X_\lambda)$ represents the conjugacy class of G_λ. Then a *relative admissible surface* for a rational chain c of *degree $n > 0$* is a continuous map $f : S \to X$ from a compact oriented surface S with a specified collection of boundary components $\partial_0 \subset \partial S$ such that $f(\partial_0)$ represents nc and $f(C) \subset X_\lambda$ for some $\lambda = \lambda(C)$ for each boundary component $C \subset \partial S$ outside of ∂_0.

Proposition 2.9 *For any rational chain $c \in C_1(G)$, we have*

$$\mathrm{scl}_{(G,\mathcal{G})}(c) = \inf \frac{-\chi^-(S)}{2n(S)},$$

where the infimum is taken over all relative admissible surfaces for c.

Proof On the one hand, any relative admissible surface S for c of degree n is admissible for a chain $c + \sum c_\lambda$ of degree n, where $n \sum c_\lambda$ is the chain represented by the boundary components of S outside of the specified components ∂_0. This proves the "$\leq$" direction.

On the other hand, consider a chain $c + \sum c_\lambda$ with all c_λ rational. Any admissible surface S for $c + \sum c_\lambda$ of degree n is a relative admissible surface for c of degree n by taking the boundary components representing nc to be the specified components ∂_0. Thus for such a rational chain, $\mathrm{scl}(c + \sum c_\lambda)$ is no less than the right-hand side of the desired equality. Then the "$\geq$" direction follows by continuity. $\square$

With appropriate homology conditions, we have $\mathrm{scl}_{(G,\mathcal{G})}(c) = \mathrm{scl}_G(c)$ for $c \in C_1(G)$ if scl_G vanishes on each $H \in \mathcal{G}$. In this case, one can compute scl_G using relative admissible surfaces, which has the advantage of not closing up additional boundary components representing chains with trivial scl. Proposition 2.11 handles the case we need for graphs of groups.

A relative admissible surface is called *extremal* if it obtains the infimum in Proposition 2.9. For scl in the absolute sense, extremal surfaces are π_1-injective [7, Proposition 2.104]. This generalizes to the relative case with the same proof using the fact that free groups are LERF.

Proposition 2.10 *If $f : S \to X$ is an extremal relative admissible surface for a rational chain $c \in B_1^H(G, \mathcal{G})$, then $f_* : \pi_1(S) \to \pi_1(X)$ is injective.*

2.3 Graphs of groups

Throughout this paper, edges on graphs are oriented. In this subsection, we consider graphs $\Gamma = (V, E)$ where E includes both orientations of edges. So we have an involution $e \mapsto \bar{e}$ on E without fixed point by reversing edge orientations, and maps $o, t : E \to V$ taking origin and terminus vertices of edges respectively, such that $t(e) = o(\bar{e})$.

Suppose we have a graph $\Gamma = (V, E)$ and two collections of groups $\{G_v\}_{v \in V}$ and $\{G_e\}_{e \in E}$ indexed by vertices and edges such that $G_e = G_{\bar{e}}$. Suppose we also have

injections $o_e : G_e \to G_{o(e)}$ and $t_e : G_e \to G_{t(e)}$ for each oriented edge e satisfying $o_{\bar{e}} = t_e$. Let X_v and $X_e = X_{\bar{e}}$ be $K(G_v, 1)$ and $K(G_e, 1)$ spaces with base points b_v and $b_e = b_{\bar{e}}$ respectively. For each edge e, the injections o_e and t_e determine (up to homotopy) maps $o_e : (X_e, b_e) \to (X_{o(e)}, b_{o(e)})$ and $t_e : (X_e, b_e) \to (X_{t(e)}, b_{t(e)})$ respectively. Let X be the space obtained from the disjoint union $(\sqcup_{v \in V} X_v) \sqcup (\sqcup_{e \in E} X_e \times [-1, 1])$ by identifying $X_e \times \{t\}$ with $X_{\bar{e} \times \{-t\}}$ and gluing $X_e \times \{-1\}$ and $X_e \times \{1\}$ to $X_{o(e)}$ and $X_{t(e)}$ via o_e and t_e, respectively.

We call X the *graph of spaces* associated to the given data. Identify X_v with its image in X, referred to as the *vertex space*. For each vertex v, the image of $X_v \cup (\cup_{t(e)=v} X_e \times [0, 1))$ is homotopic to X_v, so its completion is too; we refer to both as the *thickened vertex space $N(X_v)$*. Identify X_e with the image of $X_e \times \{0\}$, called the *edge space*.

When Γ is connected, we call $G = \pi_1(X)$ the (fundamental group of) *graph of groups* and X the *standard realization* of G. We use the notation $G = \mathcal{G}(\Gamma, \{G_v\}, \{G_e\})$ to specify the underlying data. It is a fact that G_v and G_e sit inside G as subgroups via the inclusions, referred to as *vertex groups* and *edge groups*. Graphs of groups are natural generalizations of amalgams and HNN extensions. For more details, especially their actions on trees, see [27].

We say an element $g \in G$ and its conjugacy class are *elliptic* if g conjugates into some vertex group, otherwise they are *hyperbolic*. Geometrically, g is elliptic if and only if it is represented by a loop supported in some vertex space.

Let γ be a loop in X representing the conjugacy class of an element $g \in G$. We can homotope γ so that it is either disjoint from $X_e \times \{t\}$ or intersects it only at $b_e \times \{t\}$ transversely, for all $t \in (-1, 1)$ and any edge e. Then the edge spaces cut γ into finitely many arcs, unless γ is supported in a vertex space.

Each arc a is supported in some thickened vertex space $N(X_v)$ and decomposes into three parts (see Figure 1): an arc parameterizing $b_e \times [0, 1]$, a based loop in X_v, and an arc parameterizing $b_{e'} \times [-1, 0]$, where $t(e) = v = o(e')$. We refer to the element $w(a) \in G_v$ represented by the based loop as *the winding number* of a, and denote e, e' by $e_{in}(a), e_{out}(a)$ respectively. We say γ *trivially backtracks* if for some arc a as above $\overline{e_{out}(a)} = e_{in}(a)$ and $w(\alpha)$ lies in $t_e(G_e)$. In this case, γ can be simplified by a homotopy reducing the number of arcs. See Figure 1. After finitely many simplifications, the loop γ does not trivially backtrack, which we call a *tight* loop.

In the case where g is elliptic, γ is tight if and only if it is supported in some vertex space. Moreover, instead of a collection of arcs, we have a single loop γ, whose winding number $w(\gamma)$ is only well-defined up to conjugacy.

The following proposition allows us to use relative admissible surfaces to compute scl.
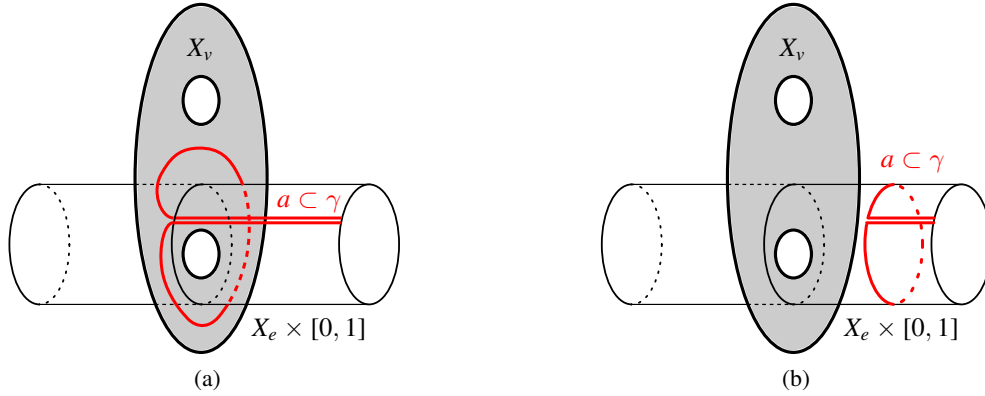
Figure 1: A loop γ trivially backtracks at an arc a supported in the thickened vertex space $N(X_v)$ as in (1a). It can be pushed off the vertex space X_v by a homotopy as in (1b).

Proposition 2.11 *Let $G = \mathcal{G}(\Gamma, \{G_v\}, \{G_e\})$ be a a graph of groups with underlying graph $\Gamma = (V, E)$. Suppose $\mathrm{scl}_G(c) = 0$ for any $c \in B_1^H(G_v)$ and vertex v. Then for any null-homologous chain $c \in B_1^H(G)$, we have*

$$\mathrm{scl}_{(G,\{G_v\})}(c) = \mathrm{scl}_G(c).$$

Proof A simple calculation shows

$$\begin{aligned} H_1(G; \mathbb{R}) &\cong H_1(\Gamma; \mathbb{R}) \\ (2\text{--}1) &\qquad \oplus \left(\bigoplus_v H_1(G_v; \mathbb{R}) \right) \Big/ \langle o_{e*}(c_e) - t_{e*}(c_e) : c_e \in H_1(G_e; \mathbb{R}) \rangle . \end{aligned}$$

Consider any null-homologous chain $c + \sum c_i$ where $g_i c_i g_i^{-1} \in C_1(G_{v_i})$ for some $g_i \in G$ and vertex v_i. It follows that $\sum c_i$ is null-homologous since c is. Thus by the homology calculation (2–1), there exist chains $c_e = -c_{\bar{e}} \in C_1(G_e)$ such that, for each vertex v, the chain

$$c_v := \sum_{i:v_i=v} c_i + \sum_{e:t(e)=v} t_e(c_e) \quad \in C_1(G_v)$$

is null-homologous in $H_1(G_v; \mathbb{R})$. Clearly, for each e the chains $t_{\bar{e}}(c_{\bar{e}}) = o_e(-c_e)$ and $-t_e(c_e)$ are equivalent to each other as they are geometrically homotopic. Hence $c + \sum c_i = c + \sum_{v \in V} c_v \in C_1^H(G)$, and

$$\mathrm{scl}_G \left(c + \sum c_i \right) = \mathrm{scl}_G \left(c + \sum c_v \right) = \mathrm{scl}_G(c),$$

where the last equality holds since $\mathrm{scl}_G(c_v) = 0$ for all v by assumption. Then the result follows from the definition of relative scl. ☐

In the sequel, all relative admissible surfaces will be understood to be relative to vertex groups unless stated otherwise.

3 Surfaces in graphs of groups

Throughout this section, let $G = \mathcal{G}(\Gamma, \{G_v\}, \{G_e\})$ be a graph of groups and let X be its standard realization as in Subsection 2.3. Let $\underline{g} = \{g_1, \dots, g_m\}$ be a set of infinite order elements in G represented by tight loops $\underline{\gamma} = \{\gamma_1, \dots, \gamma_m\}$ in X.

Recall that edge spaces cut hyperbolic tight loops into arcs, while each elliptic tight loop is already supported in some vertex space. For each vertex v, denote by A_v the collection of arcs supported in $N(X_v)$ obtained by cutting hyperbolic loops in $\underline{\gamma}$. Arcs from different loops or different parts of the same loop are considered as distinct elements. Let $L_v \subset \underline{\gamma}$ be the set of elliptic loops supported in X_v.

For each $\alpha \in A_v \cup L_v$, let $i(\alpha)$ be the index such that α sits on $\gamma_{i(\alpha)}$. Denote by $[w(\alpha)]$ the homology class of the winding number $w(\alpha)$ in $H_1(G_v; \mathbb{R})$.

Lemma 3.1 *Let* $c = \sum_i r_i g_i$ *be a null-homologous chain. Then there exists* $\beta_e \in H_1(G_e; \mathbb{R})$ *for each edge* e *such that* $\beta_e = -\beta_{\bar{e}}$ *and*

$$\sum_{\alpha \in A_v \cup L_v} r_{i(\alpha)}[w(\alpha)] = \sum_{t(e)=v} t_{e*}\beta_e$$

holds for each vertex v.

Proof This directly follows from equation (2–1). ☐

3.1 Normal form

With the above setup, fix a rational homologically trivial chain $c = \sum_i r_i g_i$ with $r_i \in \mathbb{Q}$. Let $f : S \to X$ be an arbitrary admissible surface for c without sphere components. Put S in general position so that it is transverse to all edge spaces. Then $F = f^{-1}(\cup_e X_e)$ is a proper submanifold of codimension 1, that is, a union of embedded loops and proper arcs. Eliminate all trivial loops in F (innermost first) by homotopy and then compress S along a loop in F whose image is trivial in X if any. Since this process decreases
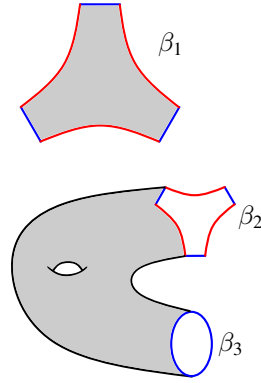
Figure 2: An example of S_v consisting of two components and three boundary components. On the boundary, the blue parts are supported in edge spaces and the red in the interior of $N(X_v)$. The boundaries β_1 and β_2 are polygonal, with arcs curly and turns straight. The boundary β_3 is a loop boundary.

$-\chi(S)$, all loops in F are non-trivial in X after finitely many repetitions. All proper arcs in F are essential since loops in $\underline{\gamma}$ are tight.

Now cut S along F into surfaces with corners, each component mapped into $N(X_v)$ for some vertex v. Let S_v be the union of components mapped into $N(X_v)$. The boundary components of S_v fall into two types (See Figure 2).

(1) Polygonal boundary: these are the boundary components divided by corners of S_v into segments alternating between proper arcs in F and arcs in A_v.

(2) Loop boundary: these are the components disjoint from corners of S_v, and thus each is a loop in either F or L_v.

Note that any disk component in S_v must bound a polygonal boundary since loop boundaries are non-trivial in X by the simplification process above and the assumption that $\underline{g}$ consists of infinite order elements.

Let α be any proper arc in F that appears on a polygonal boundary of S_v. If α with the orientation induced from S_v starts from an arc $a_v \in A_v$ and ends in $a'_v \in A_v$, we say α is a *turn* from a_v to a'_v. With the induced orientation, α represents an element $w \in G_e$, called the *winding number* of this turn, where $e = e_{out}(a_v) = \overline{e_{in}(a'_v)}$. The triple (a_v, w, a'_v) determines the *type* of the turn α. There are possibly many turns in S_v of the same type.

Suppose $u = t(e)$ is the other end of the edge e above. Then α, viewed from S_u, gives rise to a turn in S_u from a_u to a'_u, where $a_u, a'_u \in A_u$ are arcs such that a_v is followed
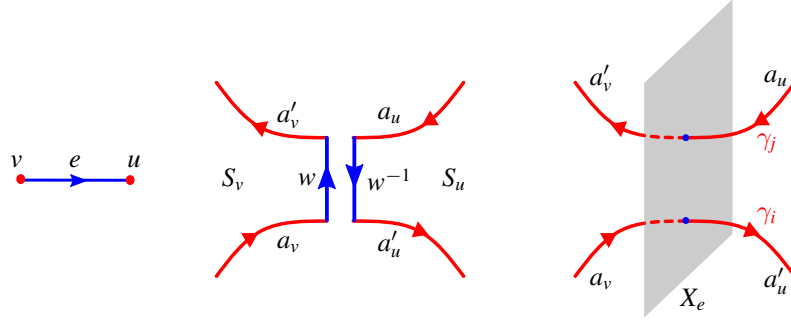
Figure 3: Two paired turns on an edge space X_e, with arcs $a_v, a'_u \subset \gamma_i$ and $a'_v, a_u \subset \gamma_j$, $e = e_{out}(a_v)$

by a'_u and a_u is followed by a'_v on γ. See Figure 3. Since the two sides induce opposite orientations on α, the winding numbers of the two turns are inverses. We refer to such two types of turns (a_v, w, a'_v) and (a_u, w^{-1}, a'_u) as *paired turns*.

Similar analysis can be done for loops in F except that they represent conjugacy classes in G_e instead of elements.

Then the collection of boundaries $\{\partial S_v\}$ satisfies the following

(3–1)   *gluing condition* : Turns of paired types have equal numbers of instances.

Moreover, the loop boundaries together represent a trivial chain in $B_1^H(G)$.

Definition 3.2 Such a decomposition into subsurfaces S_v, one for each vertex v, is called the *normal form* of S.

Recall the orbifold Euler characteristic of a surface Σ with corners is

$$\chi_o(\Sigma) := \chi(\Sigma) - \frac{1}{4}\#\text{corners}.$$

Then we obtain a surface S' admissible for a chain equivalent to c in $B_1^H(G)$ of the same degree as S by gluing up turns of paired types arbitrarily on polygonal boundaries in the normal form of S, and

$$-\chi(S') = -\sum \chi_o(S_v) \leq -\chi(S).$$

We summarize the discussion above as the following lemma.

Lemma 3.3 *Every admissible surface S can be decomposed into the normal form after throwing away sphere components, compressions, homotopy and cutting along edge spaces. By gluing up paired turns arbitrarily on polygonal boundaries in the normal form of S, we obtain a surface S′ relative admissible for c of the same degree as S and*

$$-\chi(S') = -\sum_v \chi_o(S_v) \le -\chi(S).$$

The following theorem pieces isometric embeddings of vertex groups into a global one.

Theorem 3.4 (first isometric embedding) *Let $\mathcal{G}(\Gamma, \{G_v\}, \{G_e\})$ and $\mathcal{G}(\Gamma', \{G_v'\}, \{G_e'\})$ be graphs of groups. Suppose there is a graph isomorphism $h : \Gamma \to \Gamma'$, homomorphisms $h_v : G_v \to G_{h(v)}'$ and isomorphisms $h_e : G_e \to G_{h(e)}'$ such that*

(1) *each h_v is an isometric embedding;*

(2) *for each v, the map induced by h_v on homology is injective on H_v, where H_v is the sum of $\mathrm{Im} t_{e*}$ over all edges e with $t(e) = v$; and*

(3) *the following diagram commutes for all edge pairs (e, e') with $e' = h(e)$.*

$$
\begin{array}{ccccc}
G_{o(e)} & \xleftarrow{\ o_e\ } & G_e & \xrightarrow{\ t_e\ } & G_{t(e)} \\
{\scriptstyle h_{o(e)}}\downarrow & & {\scriptstyle h_e}\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle h_{t(e)}} \\
G'_{o(e')} & \xleftarrow{\ o_{e'}\ } & G'_{e'} & \xrightarrow{\ t_{e'}\ } & G'_{t(e')}
\end{array}
$$

Then the induced homomorphism $h : \mathcal{G}(\Gamma, \{G_v\}, \{G_e\}) \to \mathcal{G}(\Gamma', \{G_v'\}, \{G_e'\})$ is an isometric embedding.

Proof It easily follows from assumptions (1) and (3) that h is well defined and injective. Now we show h preserves scl.

Consider an arbitrary rational homologically trivial chain $c = \sum_i r_i g_i$ with $r_i \in \mathbb{Q}$ and $g_i \in G$ of infinite order. Represent each g_i by a tight loop γ_i in the standard realization X of G. Define $\underline{\gamma}, A_v, L_v$ as in the discussion above.

Let X' be the standard realization of G' and let $\eta : X \to X'$ be the map representing h. Then the arcs obtained by cutting $\sqcup_i \eta(\gamma_i)$ along the edge spaces of X' are exactly $\sqcup_v \eta(A_v)$.

Let S' be an admissible surface for $c' = h(c)$ of degree n'. It suffices to show, for any $\epsilon > 0$, there is an admissible surface S for c of degree n such that

$$\frac{-\chi^-(S)}{n} \le \frac{-\chi^-(S')}{n'} + C(S')\epsilon$$

for some constant $C(S')$.

By Lemma 3.3, we may assume S' to be in its normal form and obtained by gluing surfaces with corner $S'_{v'}$, where $S'_{v'}$ is supported in the thickened vertex space $N(X'_{v'})$. Let $v = h^{-1}(v')$. Now we pull back $\partial S'_{v'}$ in a natural way and show that the pull back is homologically trivial in G_v. First consider each polygonal boundary of $S'_{v'}$. Each arc is in $\eta(A_v)$ and pulls back via h_v to a unique arc in A_v. Each turn is in $\text{Im} t_{e'}$ for some e' with $t(e') = v'$, and thus, by assumption (3), pulls back via h_v to a unique turn in $\text{Im} t_e$, where $e = h^{-1}(e')$. Similarly consider each loop boundary of $S'_{v'}$. If such a loop lies in $\eta(L_v)$, then it uniquely pulls back to a loop in L_v. Otherwise, it is obtained by cutting along the preimage of edge spaces, thus uniquely pulls back to a loop represented by an element in $\text{Im} t_e$. By construction, the homology of the pull back of $\partial S'_{v'}$ lies in

$$n' \sum_{\alpha \in A_v \cup L_v} r_{i(\alpha)}[w(\alpha)] + H_v,$$

which is exactly H_v by Lemma 3.1 since c is homologically trivial. Thus assumption (2) implies the pull back of $\partial S'_{v'}$ is homologically trivial in G_v. By assumption (1) and Lemma 2.5, there is a surface S_v (with corners induced by those on $S'_{v'}$) mapped into X_v such that $h_v(\partial S_v) = n_v \partial S'_{v'}$ for some integer $n_v > 0$ and

$$\frac{-\chi(S_v)}{n_v} \le -\chi(S'_v) + \epsilon.$$

The analogous inequality holds with χ replaced by χ_o.

Now let $N = \prod_v n_v$ and take N/n_v copies of S_v for every vertex v. These pieces satisfy the gluing condition (3–1) and can be glued to form an admissible surface for c of degree $n = N \cdot n'$ since $h_v(\partial S_v) = n_v \partial S'_{v'}$ and $\{S'_{v'}\}_{v'}$ glues up to S', which is admissible of degree n'. Note that S_v has no disk components with loop boundary, and thus S has no sphere components. Each g_i is of infinite order, so S has no disk components either. Hence $\chi^-(S) = \chi(S)$ and

$$
\begin{aligned}
\frac{-\chi(S)}{n} &= \frac{\sum_v -\chi_o(S_v) \cdot \frac{N}{n_v}}{N \cdot n'} \\
&\le \frac{\sum_{v'}[-N\chi_o(S'_{v'}) + N\epsilon]}{N \cdot n'} \\
&\le \frac{-\chi^-(S')}{n'} + \frac{\#\{v'\}}{n'} \cdot \epsilon,
\end{aligned}
$$

where the summations are taken over vertices v' where S' has nonempty intersection with $X_{v'}$ and $\#\{v'\}$ is the number of such vertices, which is finite by compactness of S'. $\qquad \square$

In the special case of free products, this is [17, Theorem B], whose applications can be found in [17, Section 5].

Corollary 3.5 Let w be an element in a group G representing a non-trivial rational homology class. Let $G(w, M, L)$ be the HNN extension $G*_{\mathbb{Z}}$ given by the inclusions $o_e, t_e : \mathbb{Z} \to G$ sending the generator 1 to w^M and w^L respectively, where $M, L \neq 0$. Then the homomorphism $h : \mathrm{BS}(M, L) \to G(w, M, L)$ is an isometric embedding.

Proof Consider both groups as graphs of groups where the underlying graph has one vertex and one edge. The inclusion of the vertex group $\mathbb{Z} = \langle a \rangle \to G$ with $a \mapsto w$ is injective since $[w] \neq 0 \in H_1(G; \mathbb{Q})$, and is an isometric embedding since $\mathbb{Z}$ is abelian (See Example 2.4). The non-triviality of $[w]$ also implies that condition (2) of Theorem 3.4 holds. Then it is easy to see that Theorem 3.4 applies. $\qquad\square$

3.2 Simple normal form

From now on, we assume $\mathrm{scl}_G(c) = 0$ for all $c \in B_1^H(G_v)$ and all vertices v, which holds for example when $\mathrm{scl}_{G_v} \equiv 0$ by monotonicity of scl. In view of Proposition 2.11 we are interested in admissible surfaces relative to vertex groups. Each such a surface is admissible (in the absolute sense) for some rational chain, and thus has a normal form possibly after simplifications. We now further simplify the normal form using the triviality of scl in the vertex groups.

Definition 3.6 A normal form of an admissible surface S relative to vertex groups is called the *simple normal form* if each component in S_v has exactly one polygonal boundary and is either a disk or an annulus with the former case happening if and only if the polygonal boundary is null-homotopic in S_v. For simplicity, we refer to such a surface as a *simple relative admissible surface*.

Assume each $g_i \in \underline{g}$ to be hyperbolic. Let $c = \sum r_i g_i$ be a rational chain whose homology class $[c]$ is in the kernel of the projection $H_1(G; \mathbb{R}) \to H_1(\Gamma; \mathbb{R})$.

Lemma 3.7 *For any relative admissible surface S for c, there is another S' of the same degree in simple normal form, such that*
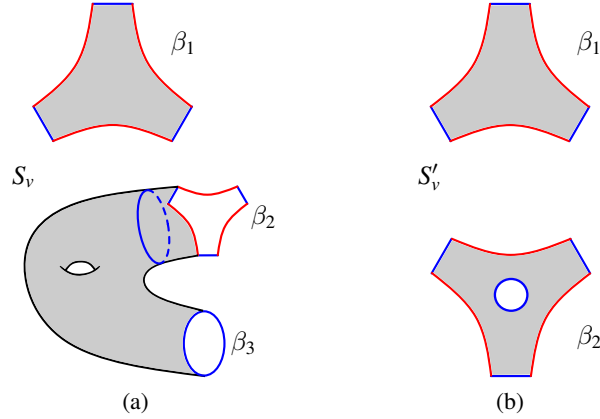
$$-\chi(S') \leq -\chi(S).$$

Figure 4: With the S_v in Figure 2, we cut out a neighborhood of β_2, shown on the left, and throw away the component without polygonal boundary to obtain S_v' on the right.

Proof By Lemma 3.3, we may assume S to be in its normal form. For each subsurface S_v, in any of its components other than disks, take out a small collar neighborhood of each polygonal boundary. Let S_v' be the disjoint union of these collar neighborhoods and disk components of S_v. See Figure 4. Then each component of the subsurfaces in S_v that we throw away to obtain S_v' has at least one boundary component, has no corners, and cannot be a disk. Thus

$$-\chi_o(S_v') \leq -\chi_o(S_v).$$

Since S_v' has all its polygonal boundaries taken from S_v, it satisfies the gluing condition (3–1). Recall that disk components of S_v must have polygonal boundaries. It follows that each component of S_v' has exactly one polygonal boundary. Moreover, each loop boundary in S_v' is homotopic to a polygonal boundary by construction, and thus to a loop in X_v. If the loop boundary is null-homotopic, replace the interior by a disk realizing the null-homotopy. Since each $g_i \in \underline{g}$ is hyperbolic, gluing up paired turns in S_v' produces a relative admissible surface S' for c of the same degree as S with the desired properties. $\square$

Theorem 3.8 (second isometric embedding) *Let* $\mathcal{G}(\Gamma, \{G_v\}, \{G_e\})$ *and* $\mathcal{G}(\Gamma', \{G_v'\}, \{G_e'\})$ *be graphs of groups where scl vanishes on vertex groups. Suppose there is a graph homomorphism* $h : \Gamma \to \Gamma'$*, injective homomorphisms* $h_v : G_v \to G_{h(v)}'$ *and isomorphisms* $h_e : G_e \to G_{h(e)}'$ *such that*

(1) *h is injective on the set of edges;*

(2) *the map induced by h_v on homology is injective on H_v, where H_v is the sum of Imt_{e*} over all edges e with $t(e) = v$; and*

(3) *the following diagram commutes for all edge pairs (e, e') with $e' = h(e)$.*

$$
\begin{array}{ccccc}
G_{o(e)} & \xleftarrow{\ o_e\ } & G_e & \xrightarrow{\ t_e\ } & G_{t(e)} \\
h_{o(e)} \downarrow & & h_e \downarrow \cong & & h_{t(e)} \downarrow \\
G'_{o(e')} & \xleftarrow{\ o_{e'}\ } & G'_{e'} & \xrightarrow{\ t_{e'}\ } & G'_{t(e')}
\end{array}
$$

Then the induced homomorphism $h : \mathcal{G}(\Gamma, \{G_v\}, \{G_e\}) \to \mathcal{G}(\Gamma', \{G'_v\}, \{G'_e\})$ is an isometric embedding.

Proof The proof is similar to that of Theorem 3.4 by considering an arbitrary relative admissible surface S' for $c' = h(c)$ in simple normal form instead of normal form, where $c = \sum r_i g_i$ is a rational chain in G with all g_i hyperbolic. Elliptic elements are ignored since it suffices to show that h preserves scl relative to vertex groups according to Proposition 2.11. The difference is that, the graph homomorphism h is now allowed to collapse vertices, so there might be several vertices of Γ mapped to the same vertex v' in Γ'. However, each component of $S'_{v'}$ has a polygonal boundary on which each turn is supported in some $X'_{e'}$ connecting two arcs in $A'_{v'}$, where $t(e') = v'$. Then the two arcs must be the image of two arcs in A_v under h_v for the vertex $v = t(e) \in h^{-1}(v)$, where $e = h^{-1}(e')$. It follows that all the arcs on each polygonal boundary come from the same vertex v and thus each polygonal boundary of $S'_{v'}$ can be pulled back to a polygonal boundary in a unique X_v with $h(v) = v'$. Then the rest of the proof is the same as that of Theorem 3.4. $\qquad\qquad\square$

An immediate corollary is the following proposition, which is useful to simplify the computation of scl.

Proposition 3.9 (restriction of domain) *Let $\mathcal{G}(\Gamma, \{G_v\}, \{G_e\})$ be a graph of groups where scl vanishes on vertex groups. Let Γ' be a subgraph of Γ, then the inclusion $i : \mathcal{G}(\Gamma', \{G_v\}, \{G_e\}) \to \mathcal{G}(\Gamma, \{G_v\}, \{G_e\})$ is an isometric embedding.*

We can use Theorem 3.8 to compute the scl of t-alternating words in HNN extensions of abelian groups.

Definition 3.10 Given injective homomorphisms $i, j : E \to V$, let $G = V *_E = \langle G, t \mid i(e) = t\, j(e)t^{-1} \rangle$ be the HNN extension. An element g is *t-alternating* if it is conjugate to a cyclically reduced word of the form $a_1 t b_1 t^{-1} \ldots a_n t b_n t^{-1}$ for some $n \geq 0$.

Let $H = V_1 *_E V_2$ be the amalgam with $V_1 = V_2 = V$ and injections i, j above, which has a natural inclusion into G, whose image is the set of t-alternating words.

Proposition 3.11 (t-alternating words) *Let V and E be abelian groups with inclusions $i, j : E \to V$, then we have an isometric embedding $h : V_1 *_E V_2 \to V*_E$ where $V_1 = V_2 = V$. In particular, with $T = t^{-1}$, we have*

$$
\begin{aligned}
\mathrm{scl}_{(V*_E, V)}(a_1 t b_1 T a_2 t b_2 T \cdots a_n t b_n T) &= \mathrm{scl}_{(V_1 *_E V_2, \{V_1, V_2\})}(a_1 b_1 \cdots a_n b_n) \\
&= \mathrm{scl}_{(V_1/i(E) * V_2/j(E), \{V_1/i(E), V_2/j(E)\})}(\bar{a}_1 \bar{b}_1 \cdots \bar{a}_n \bar{b}_n).
\end{aligned}
$$

Proof Consider $V_1 *_E V_2$ and $V*_E$ as graphs of groups where the underlying graphs are a segment and a loop respectively. Let h be the graph homomorphism taking the two end points of the segment to the vertex on the loop. Let h_v and h_e be identities. It is easy to check that the assumptions of Theorem 3.8 are satisfied since V and E are abelian. Thus we obtain an induced isometric embedding $h : V_1 *_E V_2 \to V*_E$ with $h(a_1 b_1 \cdots a_n b_n) = a_1 t b_1 T a_2 t b_2 T \cdots a_n t b_n T$.

Let $a_0 \in V_1$ and $b_0 \in V_2$ be elements such that the chain $c = a_0 + b_0 + a_1 b_1 \cdots a_n b_n$ is null-homologous. Then by Proposition 2.11, the isometric embedding h provides

$$
\begin{aligned}
\mathrm{scl}_{(V_1 *_E V_2, \{V_1, V_2\})}(a_1 b_1 \cdots a_n b_n) &= \mathrm{scl}_{V_1 *_E V_2}(a_0 + b_0 + a_1 b_1 \cdots a_n b_n) \\
&= \mathrm{scl}_{V*_E}(a_0 + b_0 + a_1 t b_1 T a_2 t b_2 T \cdots a_n t b_n T) \\
&= \mathrm{scl}_{(V*_E, V)}(a_1 t b_1 T a_2 t b_2 T \cdots a_n t b_n T)
\end{aligned}
$$

The other equality can be proved similarly using the scl-preserving projection π [28, Proposition 4.3] in the central extension

$$
1 \to E \to V_1 *_E V_2 \xrightarrow{\pi} V_1/i(E) * V_2/j(E) \to 1.
$$

$\square$

In particular, scl of t-alternating words in Baumslag–Solitar groups can be computed as scl in free products of cyclic groups, which is better understood and easier to compute.

Corollary 3.12 Let $\mathbb{Z}/M\mathbb{Z} * \mathbb{Z}/L\mathbb{Z} = \langle x, y \mid x^M = y^L = 1 \rangle$. For any t-alternating word in $\mathrm{BS}(M, L)$, we have

$$
\mathrm{scl}_{\mathrm{BS}(M,L)}(a^{u_1} t a^{v_1} t^{-1} a^{u_2} t a^{v_2} t^{-1} \cdots a^{u_n} t a^{v_n} t^{-1}) = \mathrm{scl}_{\mathbb{Z}/M\mathbb{Z} * \mathbb{Z}/L\mathbb{Z}}(x^{u_1} y^{v_1} \cdots x^{u_n} y^{v_n}).
$$

Proof It directly follows from Proposition 2.11 and Proposition 3.11. $\square$

The scl spectrum of a group G is the image of scl_G (as a function on G). Corollary 3.12 implies that the spectrum of $\mathbb{Z}/M\mathbb{Z} * \mathbb{Z}/L\mathbb{Z}$ is a subset of the spectrum of $\mathrm{BS}(M, L)$. Due to our limited understanding of the scl spectrum, it is not clear if this is a proper subset and how big the difference is. We do know that the smallest positive elements in the two spectra exist and agree if M and L are odd; See [19, Corollary 5.10] and [18, Remark 3.6].

Here are a few examples, where we write $T = t^{-1}$ and $A = a^{-1}$ for simplicity.

Example 3.13 With the notation in Corollary 3.12, the product formula [7, Theorem 2.93] implies

$$\mathrm{scl}_{\mathrm{BS}(M,L)}(atAT) = \mathrm{scl}_{\mathbb{Z}/M*\mathbb{Z}/L}(xy^{-1}) = \frac{1}{2}\left(1 - \frac{1}{M} - \frac{1}{L}\right).$$

This explains why [20, Proposition 5.5] resembles the product formula.

Example 3.14 Similarly, using [17, Proposition 5.6] instead of the product formula, we have

$$\mathrm{scl}_{\mathrm{BS}(M,L)}(ataTAtAT) = \mathrm{scl}_{\mathbb{Z}/M*\mathbb{Z}/L}([x, y]) = \frac{1}{2} - \frac{1}{\min(M, L)}.$$

Example 3.15 Finally, for explicit M and L, one can use Corollary 3.12 and the computer program `scallop` [14] to quickly compute scl of rather long t-alternating words. For example,

$$\mathrm{scl}_{\mathrm{BS}(7,5)}(ataTataTatATatATataTatATatATatATataT) = \frac{123}{70}$$

$$\mathrm{scl}_{\mathrm{BS}(5,11)}(ataTataTatATatATataTataTatATatATatATataTataTatAT) = \frac{102}{55}.$$

Proposition 3.11 also provides a new perspective and a shorter proof for [12, Proposition 4.4] as follows.

Proposition 3.16 (Calegari–Walker [12]) *The homomorphism*

$$h : F_2 = \langle x, y \rangle \rightarrow F_2 = \langle a, t \rangle$$

given by $h(x) = a$ and $h(y) = tat^{-1}$ is an isometric embedding.

Proof Consider the domain F_2 as a free product and the co-domain F_2 as a free HNN extension of $\mathbb{Z}$. Then the conclusion immediately follows from Proposition 3.11 and Proposition 2.11. □

4 Asymptotic promotion

In the sequel, we consider graphs of groups $\mathcal{G}(\Gamma, \{G_v\}, \{G_e\})$ where

(1) $\mathrm{scl}_{G_v} \equiv 0$ for each vertex v, and

(2) the images of edge groups in each vertex group are *central* and *mutually commensurable*.

The first assumption can be weakened to $\mathrm{scl}_G(c) = 0$ for any $c \in B_1^H(G_v)$ and any v, but it is usually hard to check without having $\mathrm{scl}_{G_v} \equiv 0$.

Example 4.1 The following families of groups satisfy both assumptions above.

(1) Any graphs of groups with vertex and edge groups isomorphic to $\mathbb{Z}$, since all non-trivial subgroups of $\mathbb{Z}$ are commensurable. These groups are also known as generalized Baumslag–Solitar groups.

(2) Amalgams of abelian groups, since the commensurability assumption is vacuous for degree-one vertices. This includes the groups studied in [28].

(3) Graphs of groups where vertex groups are isomorphic to the Heisenberg group $\mathcal{H}_3(\mathbb{Z})$ or fundamental groups of irreducible 3-manifolds with Nil geometry, and edge groups are isomorphic to $\mathbb{Z}$ and maps into the central $\mathbb{Z}$ subgroup of the vertex groups generated by a regular Seifert fiber (see [26]). Note that these vertex groups are amenable (actually virtually solvable [30, Theorem 4.7.8]), and thus have trivial scl.

(4) Free products of groups with trivial scl. These are the groups studied in [17].

As in the previous section, let $\underline{g}$ be a finite collection of hyperbolic elements and $\underline{\gamma}$ be their tight loop representatives. We would like to compute scl of a rational chain $c = \sum r_i g_i$ relative to vertex groups. It comes down to understanding components that might appear in simple relative admissible surfaces. The essential difficulty is that there are too many possible components since the winding number of each turn could have infinitely many choices when some edge group is infinite. One can simply ignore the winding numbers of turns to estimate the Euler characteristic to get lower bounds of scl. Clay–Forester–Louwsma [20] and Susse [28] show such a lower bound turns out to be equality in certain cases. In general, we cannot completely ignore the winding number.

Surprisingly, it turns out that recording winding numbers of turns mod certain *finite index* subgroup of edge groups is sufficient (Lemma 4.15) to asymptotically recover the surface by adjusting the winding numbers. This is the goal of this section and is the

heart of the rationality theorem (Theorem 5.10). To this end, we investigate adjustment of winding numbers in terms of *transition maps* and *adjustment maps* which we will define.

For the moment, we will also assume Γ to be *locally finite* for convenience. We can always reduce the situation to this case using restriction of domain (Proposition 3.9).

For each vertex v, let W_v be the subgroup generated by the images of adjacent edge groups. By our assumption and local finiteness, W_v is *central* in G_v and each adjacent edge group is *finite index* in W_v.

Definition 4.2 A *virtual isomorphism* $\phi : H \to H'$ is an isomorphism $\phi : H_0 \to H'_0$ of finite index subgroups $H_0 \leq H$ and $H'_0 \leq H'$. The domain $\mathrm{Dom}\phi = H_0$ and the image $\mathrm{Im}\phi = H'_0$ are part of the data of ϕ. Typically ϕ is not defined for elements outside $\mathrm{Dom}\phi$. In the case $H = H'$, we say ϕ is a *virtual automorphism*.

Two virtual isomorphisms $\phi : H_1 \to H_2$ and $\psi : H_2 \to H_3$ form a composition $\psi\phi$ with $\mathrm{Dom}\psi\phi = \phi^{-1}(\mathrm{Im}\phi \cap \mathrm{Dom}\psi)$ and $\mathrm{Im}\psi\phi = \psi(\mathrm{Im}\phi \cap \mathrm{Dom}\psi)$, which is again a virtual isomorphism. Each virtual isomorphism has an inverse in the obvious way.

A similar notion appears in [29], which agrees with ours in the context of finitely generated abelian groups.

It follows from the commensurability assumption and local finiteness of Γ that the inclusion $t_e = o_{\bar{e}} : G_e \to W_v$ is a virtual isomorphism for all edges e with $t(e) = v$. For each edge e connecting vertices $u = o(e)$ and $v = t(e)$ (possibly $u = v$), we form a *transition map* $\tau_e : W_u \to W_v$ via $\tau_e := -t_e \circ o_e^{-1}$, which is a virtual isomorphism. The negative sign makes sense since W_u and W_v are abelian, and the reason we add it will be clear when we define adjustment maps in Subsection 4.2.

Moreover, for any oriented path $P = (v_0, e_1, v_1, \ldots, e_n, v_n)$ in Γ with $o(e_i) = v_{i-1}$ and $t(e_i) = v_i$, we have a virtually isomorphic *transition map* $\tau_P : W_{v_0} \to W_{v_n}$ defined as

$$\tau_P := \tau_{e_n} \circ \cdots \circ \tau_{e_1}.$$

For any arc $a_v \in A_v$ on $\gamma_i \in \underline{\gamma}$, the loop γ_i projects to an oriented cycle $P(a_v) = (v, e_1, v_1, \ldots, e_n, v)$ in Γ, and thus gives rise to a transition map $\tau_{P(a_v)} : W_v \to W_v$ as above.

There is a stability result of virtual automorphisms that is important for our argument.

4.1 Stability of virtual automorphisms

Let $\phi : H \to H$ be a virtual automorphism. Typically both $\text{Dom}\phi^p$ and $\text{Im}\phi^p$ will keep getting smaller as $p \to \infty$. However, when H is an abelian group, the subgroup $\text{Dom}\phi^p + \text{Im}\phi^q$ generated by $\text{Dom}\phi^p$ and $\text{Im}\phi^q$ will stabilize to a finite index subgroup.

Lemma 4.3 (Stability) *Let H be an abelian group with a virtual automorphism ϕ. There is a finite index subgroup H_0 of H such that $H_0 \subset \text{Dom}\phi^p + \text{Im}\phi^q$ for any $p, q \geq 0$.*

We first look at the simplest example with $H = \mathbb{Z}$, where everything is quite explicit.

Example 4.4 Let X, Y be non-zero integers. Let $\phi : \mathbb{Z} \to \mathbb{Z}$ be the virtual automorphism given by $\phi(X) = Y$ with $\text{Dom}\phi = X\mathbb{Z}$ and $\text{Im}\phi = Y\mathbb{Z}$. Let $d = \gcd(|X|, |Y|)$, $x = |X|/d$ and $y = |Y|/d$. Then $\text{Dom}\phi^2 = \phi^{-1}(\text{Im}\phi \cap \text{Dom}\phi) = \phi^{-1}(dxy\mathbb{Z}) = dx^2\mathbb{Z}$ and $\text{Im}\phi^2 = \phi(\text{Im}\phi \cap \text{Dom}\phi) = \phi(dxy\mathbb{Z}) = dy^2\mathbb{Z}$.

More generally, we have $\text{Dom}\phi^p = dx^p\mathbb{Z}$ and $\text{Im}\phi^q = dy^q\mathbb{Z}$, both keep getting smaller as $p, q \to \infty$. However, $\text{Dom}\phi^p + \text{Im}\phi^q = \gcd(dx^q, dy^q)\mathbb{Z} = d\mathbb{Z}$ for all $p, q \geq 1$.

We prove Lemma 4.3 by computing the index. Denote the index of $B \leq A$ by $|A : B|$. Recall the following basic identities, which we will use.

Lemma 4.5 *Let A be an abelian group with finite index subgroups B and C.*

(1) *If C is a subgroup of B, then $|B : C| = |A : C|/|A : B|$.*

(2)
$$|A : B + C| = \frac{|A : B|}{|B + C : B|} = \frac{|A : B|}{|C : B \cap C|}.$$

(3) *If ϕ is an injective homomorphism defined on A, then $|A : B| = |\phi(A) : \phi(B)|$.*

Lemma 4.6 *Let H be an abelian group with a virtual automorphism ϕ. Let $I_p := |H : \text{Im}\phi^p|$.*

(1) *We have*
$$|H : \text{Dom}\phi^p + \text{Im}\phi^q| = \frac{I_p I_q}{I_{p+q}}.$$

(2) *There exist integers $r \geq 1$ and $N_\phi \geq 1$ such that $I_{q+1}/I_q = r$ for all $q \geq N_\phi$.*

Proof Note that ϕ^p isomorphically maps $\mathrm{Dom}\phi^p \cap \mathrm{Im}\phi^q$ to $\mathrm{Im}\phi^{p+q}$ and $\mathrm{Dom}\phi^p$ to $\mathrm{Im}\phi^p$. By the formulas in Lemma 4.5, we have

$$|H : \mathrm{Dom}\phi^p + \mathrm{Im}\phi^q| = \frac{|H : \mathrm{Im}\phi^q|}{|\mathrm{Dom}\phi^p : \mathrm{Dom}\phi^p \cap \mathrm{Im}\phi^q|} = \frac{I_p I_q}{I_{p+q}}.$$

To prove the second assertion, let $p = 1$ in the equation above. We have

$$\frac{I_{q+1}}{I_q} = \frac{I_1}{|H : \mathrm{Dom}\phi + \mathrm{Im}\phi^q|}.$$

Since the sequence of *integers* $|H : \mathrm{Dom}\phi + \mathrm{Im}\phi^q|$ is increasing in q with upper bound $|H : \mathrm{Dom}\phi|$, it must stabilize when $q \geq N_\phi$ for some $N_\phi \geq 1$ and thus I_{q+1}/I_q stabilizes to some r when $q \geq N_\phi$. $\qquad\square$

Proof of Lemma 4.3 With notation as in Lemma 4.6, we have $I_q = r^{q-N_\phi} I_{N_\phi}$ for all $q \geq N_\phi$. Thus

$$|H : \mathrm{Dom}\phi^p + \mathrm{Im}\phi^q| = \frac{I_p I_q}{I_{p+q}} = \frac{r^{p+q-2N_\phi} I_{N_\phi}^2}{r^{p+q-N_\phi} I_{N_\phi}} = \frac{I_{N_\phi}}{r^{N_\phi}},$$

for any $p, q \geq N_\phi$.

Since $\mathrm{Dom}\phi^p + \mathrm{Im}\phi^q \subset \mathrm{Dom}\phi^{p'} + \mathrm{Im}\phi^{q'}$ whenever $p \geq p'$ and $q' \geq q$, the index computation above shows that there is an index I_{N_ϕ}/r^{N_ϕ} subgroup H_0 of H such that $\mathrm{Dom}\phi^p + \mathrm{Im}\phi^q = H_0$ when $p, q \geq N_\phi$. Therefore, for arbitrary $p, q \geq 0$,

$$\mathrm{Dom}\phi^p + \mathrm{Im}\phi^q \supseteq \mathrm{Dom}\phi^{\max\{p,N_\phi\}} + \mathrm{Im}\phi^{\max\{q,N_\phi\}} = H_0.$$

$\qquad\square$

4.2   Disk-like pieces

Recall that any arc $a_v \in A_v$ on $\gamma_i \in \underline{\gamma}$, determines an oriented cycle $P(a_v) = (v, e_1, v_1, \ldots, e_n, v)$ in Γ, and consequently a transition map $\tau_{P(a_v)}$ which is a virtual automorphism of W_v.

Consider a simple relative admissible surface S for c. Refer to a component of a subsurface S_v as *a piece*. Recall that S is obtained by gluing pieces together along paired turns on the polygonal boundaries in a certain way that can be encoded by a graph Γ_S, where each vertex corresponds to a piece and each edge corresponds to two paired turns glued together in S. There is a graph homomorphism $\pi : \Gamma_S \to \Gamma$ taking

a vertex $\hat{v}$ to v if $\hat{v}$ corresponds to a piece C in S_v. In this way, S admits an induced structure of a graph of spaces with underlying graph Γ_S.

Recall that each piece C in S_v has a unique polygonal boundary, which as a loop in X_v represents a conjugacy class $w(C)$ in G_v called *the winding number* of C. Recall that W_v is the subgroup generated by $t_e(G_e)$ with $t(e) = v$, which is central, so it makes sense to say whether $w(C)$ lies in W_v.

Definition 4.7 We say a piece C in S_v is a *potential disk* if its winding number $w(C)$ lies in W_v.

Since S is in simple normal form, a piece C is a disk if and only if the winding number $w(C)$ is trivial. Then potential disks are exactly the pieces that can be made into disks by adjusting winding numbers of turns on their polygonal boundaries. However, this typically cannot be done for all potential disks *simultaneously*. Our remedy is to find a class of potential disks, called *disk-like pieces*, that can be made into disks simultaneously in an asymptotic sense. Moreover, we will make sure that keeping track of a finite amount of information suffices to tell whether a piece is disk-like.

Recall that each oriented edge $\hat{e}$ of Γ_S going from $\hat{v}$ to $\hat{u}$ represents a turn shared by the pieces represented by $\hat{v}$ and $\hat{u}$. Changing the winding number of the turn by $n \in G_e$ would adjust $w(\hat{v})$ and $w(\hat{u})$ by $o_e(n)$ and $-t_e(n)$ respectively, where e is the projection of $\hat{e}$ in Γ. Here the negative sign is due to the opposite orientations on the turn induced by the two pieces, which is why we add a negative sign in the definition of transition maps.

For an oriented path $\hat{P} = (\hat{v}_0, \hat{e}_1, \hat{v}_1, \ldots, \hat{e}_n, \hat{v}_n)$ in Γ_S projecting to a path P in Γ, associate to $\hat{P}$ the adjustment map $\alpha(\hat{P}) := \tau_P$. Then for any $x \in \mathrm{Dom}\,\alpha(\hat{P})$, we can adjust the winding numbers of the turns represented by $\hat{e}_i$ ($1 \le i \le n$) such that, for any $1 \le i \le n-1$, the changes to the winding number of $\hat{v}_i$ contributed by the adjustment on $\hat{e}_i$ and $\hat{e}_{i+1}$ cancel each other, and the net result of the adjustment is

(1) the winding numbers of $\hat{v}_0$ and $\hat{v}_n$ increase by x and $\alpha(\hat{P})(x)$ respectively;

(2) the winding number of $\hat{v}_i$ stays unchanged for all $1 \le i \le n-1$.

We say such an adjustment is supported on $\hat{v}_0$ and $\hat{v}_n$.

Definition 4.8 For each $a_v \in A_v$, fix a finite index subgroup $W(a_v)$ of W_v such that

$$(4\text{--}1) \qquad W(a_v) \subset \mathrm{Dom}\,\tau^p_{P(a_v)} + \mathrm{Im}\,\tau^q_{P(a_v)} \quad \text{for any } p, q \ge 0.$$
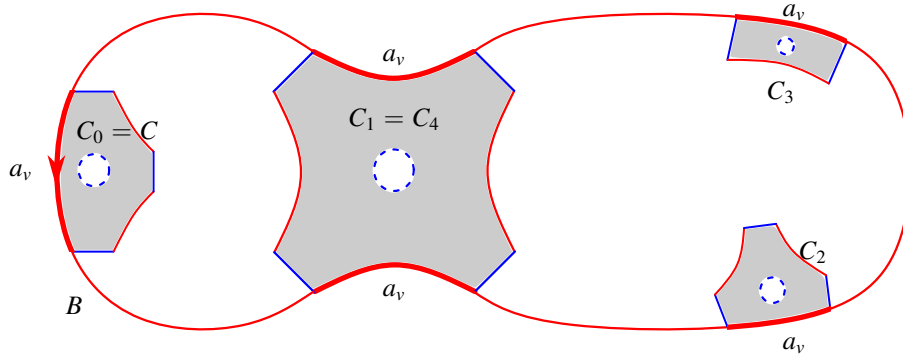
Such a $W(a_v)$ exists by Lemma 4.3.

Figure 5: An example of a boundary component B of S with $n = 5$ copies of a_v, where C_1 coincides with C_4, and where pieces sitting along B not witnessing a_v are omitted

Consider any piece C with a copy of a_v on its polygonal boundary. If $\gamma_i \in \gamma$ is the loop contain a_v, then the component B of ∂S which this copy of a_v sits on is a finite cover of γ_i, say of degree n. Then B successively passes through n copies of a_v contained in pieces C_j, $j = 0, \ldots, n - 1$, where $C_0 = C$. See Figure 5. It might happen that $C_j = C_k$ for $j \neq k$, in which case its polygonal boundary contain the j-,k-th copies of a_v on B as distinct arcs.

The boundary component B visits a sequence of pieces and thus determines an oriented cycle ω in Γ_S, on which we have vertices $\hat{v}_j$ corresponding to C_j for $j = 0, \ldots, n-1$. Let ω_j and ω_j' be the subpaths on ω going from $\hat{v}_0$ to $\hat{v}_j$ in the positive and negative orientation respectively, such that under the projection $\pi : \Gamma_S \to \Gamma$, we have $\pi(\omega_j) = P(a_v)^j$ and $\pi(\omega_j') = \overline{P(a_v)}^{n-j}$. Thus the adjustment maps are $\alpha(\omega_j) = \tau_{P(a_v)}^j$ and $\alpha(\omega_j') = \tau_{P(a_v)}^{j-n}$.

Lemma 4.9 *With the above notation, let $C_j \neq C_0$ be a piece with $w(C_j) \in W(a_v)$. Then there is an adjustment of S supported on C_0 and C_j after which C_j has trivial winding number.*

Proof By definition, we have $w(C_j) \in \mathrm{Dom}\tau_{P(a_v)}^{n-j} + \mathrm{Im}\tau_{P(a_v)}^j = \mathrm{Im}\alpha(\omega_j') + \mathrm{Im}\alpha(\omega_j)$. As a consequence, there exist $x, y \in W_v$ with $w(C_j) = \alpha(\omega_j')(x) + \alpha(\omega_j)(y)$. This gives rise to an adjustment that eliminates $w(C_j)$ and adjusts $w(C_0)$ to $w(C_0) - x - y$ without changing all other winding numbers. $\qquad\square$

Let $\widetilde{\Gamma}_S$ be a finite cover of Γ_S. This determines a cover $\widetilde{S}$ of S in simple normal form. Let $\widetilde{B}$ be a lift of B of degree m, which corresponds to a degree m lift $\widetilde{\omega}$ of ω in $\widetilde{\Gamma}_S$. Denote the m (distinct) lifts of C by $\widetilde{C}_0, \ldots, \widetilde{C}_{m-1}$ as vertices on $\widetilde{\omega}$.
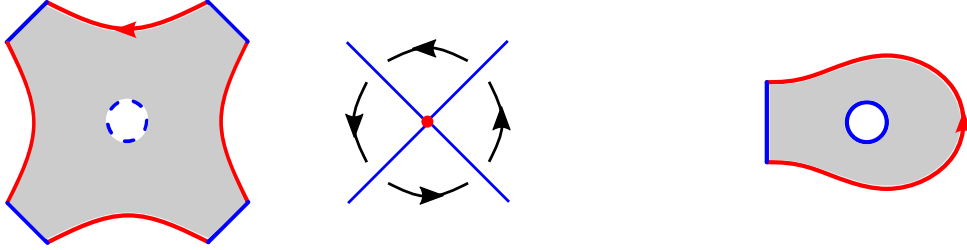
Figure 6: The cyclic order on the edges around a vertex in Γ_S (left); A piece representing a vertex where the cycle ω backtracks (right)

Lemma 4.10 *With the above notation, suppose $w(C) \in W(a_v)$, then there is an adjustment of $\widetilde{S}$ supported on $\widetilde{C}_0, \ldots, \widetilde{C}_{m-1}$ after which $\widetilde{C}_k$ has trivial winding number for all $1 \leq k \leq m-1$.*

Proof For each $1 \leq k \leq m-1$, apply Lemma 4.9 to the loop $\widetilde{\omega}$ to eliminate the winding number $w(\widetilde{C}_k)$ at the cost of changing $w(\widetilde{C}_0)$. $\qquad\square$

Once we normalize by the degree of admissible surfaces, the adjustment in Lemma 4.10 has the effect of making $1 - 1/m$ portion of an annulus piece C into disk pieces. This implies that C can be asymptotically promoted to a disk as $m \to \infty$ without affecting other pieces if $w(C) \in W(a_v)$. In the exceptional case where ω is null-homotopic in Γ_S, we cannot find finite covers with $m \to \infty$ and need a different strategy to promote C.

Lemma 4.11 *The cycle ω representing the boundary component B of S in Γ_S backtracks at a vertex $\hat{u}$ if and only if $\hat{u}$ has valence one. In this case, the piece $\hat{u}$ has only one turn on the polygonal boundary and is not a potential disk.*

Proof Let C' be a piece represented by a vertex $\hat{u}$ in Γ_S that ω passes through. Note that the edges adjacent to $\hat{u}$ correspond to the turns on the polygonal boundary of C' and thus have an induced cyclic order. See Figure 6. Since ω represents the boundary component, the two edges that ω enters and leaves $\hat{u}$ are adjacent in the cyclic order. Thus ω backtracks at $\hat{u}$ if and only if $\hat{u}$ has valence one, in which case the polygonal boundary of C' consists of one arc $a'_u \in A_u$ and one turn. Let $\hat{e}$ be the edge representing the gluing of this turn and let $e = \pi(\hat{e})$ so that $t(e) = \pi(\hat{u})$. Then $w(a'_u) \notin t_e(G_e)$ since $\underline{\gamma}$ consists of tight loops. It follows that C' cannot be a potential disk. $\qquad\square$

Lemma 4.12 *With the notation above, let B be the boundary component of S passing through a copy of a_v on a potential disk C. If the loop ω representing B in Γ_S is null-homotopic, then $P(a_v)$ is a null-homotopic loop in Γ and $\tau_{P(a_v)}$ is the identity on its domain. In particular, we have $W(a_v) \subset \mathrm{Dom}\tau_{P(a_v)}$.*

Proof The conclusions follow immediately from the fact that $P(a_v)$ is the image of ω under the graph homomorphism $\pi : \Gamma_S \to \Gamma$. We deduce $W(a_v) \subset \mathrm{Dom}\tau_{P(a_v)}$ from equation (4–1). □

Lemma 4.13 *With the notation above, suppose ω is null-homotopic and $w(C) \in W(a_v)$. Then there is an adjustment of S supported on C and a piece C' that is not a potential disk, such that $w(C)$ becomes 0 after the adjustment.*

Proof Let C' be a piece as in Lemma 4.11 representing a vertex $\hat{u}$ where ω backtracks. Let ω_0 be the positively oriented subpath of ω going from C to C'. Considering ω as a cycle based at C, we have $\mathrm{Dom}\alpha(\omega) \subset \mathrm{Dom}\alpha(\omega_0)$ for the adjustment maps. Thus $w(C) \in W(a_v) \subset \mathrm{Dom}\alpha(\omega_0)$ by Lemma 4.12 and the claimed adjustment exists. □

Definition 4.14 Given the choices of $W(a_v)$ for all arcs $a_v \in A_v$, a piece C of S_v is *a disk-like piece* if there is an arc $a_v \in A_v$ on the polygonal boundary of C such that $w(C) \in W(a_v)$. Let

$$\widehat{\chi}(C) := \left\{ \begin{array}{ll} 1 & \text{if C is disk-like,} \\ 0 & \text{otherwise} \end{array} \right.$$

be the *over-counting Euler characteristic* that counts disk-like pieces as disks. Let $\widehat{\chi}_o(C) := -\frac{1}{4}\#\text{corners} + \widehat{\chi}(C)$ be the *over-counting orbifold Euler characteristic*. For a simple relative admissible surface S, define its *over-counting Euler characteristic* as

$$\widehat{\chi}(S) := \sum \widehat{\chi}_o(C),$$

where the sum is taken over all pieces C. Equivalently,

$$\widehat{\chi}(S) = \chi(S) + \#\{C : \text{disk-like but not a disk}\}.$$

Now we show that the over-counting is accurate via asymptotic promotion.

Lemma 4.15 (Asymptotic Promotion) *Let G be a graph of groups $\mathcal{G}(\Gamma, \{G_v\}, \{G_e\})$ where*

(1) Γ *is locally finite,*

(2) $\mathrm{scl}_{G_v} \equiv 0$, *and*

(3) *the images of edge groups in each vertex group are central and mutually commensurable.*

Let $\underline{g}$ be a finite collection of hyperbolic elements of G. With the notion of disk-like pieces and $\widehat{\chi}$ above (which depends on the choices of $W(a_v)$), we have the following: For any simple relative admissible surface S for a rational chain $c = \sum r_i g_i$ of degree n, and for any $\epsilon > 0$, there is a simple relative admissible surface S' of a certain degree n' such that

$$\frac{-\chi(S')}{2n'} \leq \frac{-\widehat{\chi}(S)}{2n} + \epsilon.$$

Proof   For each disk-like piece C, say a component of S_v, fix an arc $a_v(C) \in A_v$ on the polygonal boundary of C such that $w(C) \in W(a_v(C))$. Let $\omega(C)$ be the oriented cycle in Γ_S determined by the boundary component of S containing the copy of $a_v(C)$ on ∂C.

For those pieces C with null-homotopic $\omega(C)$, by applying the adjustment as in Lemma 4.13, we assume $w(C) = 0$ and C is a genuine disk.

For any given $\epsilon > 0$, choose a large integer $N > \frac{1}{2n\epsilon}\#\{C : \text{disk-like and } w(C) \neq 0\}$. For the finite collection of non-trivial loops $\Omega := \{\omega(C) : C \text{ disk-like and } w(C) \neq 0\}$, since free groups are residually finite, there is a finite cover $\widetilde{\Gamma}_S$ of Γ_S such that any lift of each $\omega \in \Omega$ has degree at least N. Let M be the covering degree.

Let $\widetilde{S}$ be the finite cover of S corresponding to $\widetilde{\Gamma}_S$. Applying the adjustment in Lemma 4.10 to each lift of each $\omega \in \Omega$, we observe that, for every disk-like piece C with $w(C) \neq 0$, at least $M(1 - 1/N)$ of its M preimages in $\widetilde{S}$ after adjustment bounds a disk.

Denote by S' the surface obtained by the adjustment above from $\widetilde{S}$. Then S' is simple relative admissible of degree $n' = Mn$. We have

$$
\begin{aligned}
\frac{-\chi(S')}{2n'} &\leq \frac{-\chi(\widetilde{S}) - M(1 - 1/N)\#\{C : \text{disk-like and } w(C) \neq 0\}}{2nM} \\
&= \frac{-\chi(S) - (1 - 1/N)\#\{C : \text{disk-like and } w(C) \neq 0\}}{2n} \\
&= \frac{-\widehat{\chi}(S)}{2n} + \frac{\#\{C : \text{disk-like and } w(C) \neq 0\}}{2nN} \\
&< \frac{-\widehat{\chi}(S)}{2n} + \epsilon.
\end{aligned}
$$

$\square$

With Lemma 4.15 above, we can work with disk-like pieces and $\widehat{\chi}$ instead of genuine disks and χ. The advantage is that deciding whether a piece is disk-like or not is equivalent to checking whether an element in a finite abelian group vanishes or not, which only requires keeping track of a finite amount of information.

5 Determining scl by linear programming

With the help of Lemma 4.15, there are several known methods of encoding to compute scl via linear programming [5, 2, 20, 6, 32]. In this section, we use an encoding similar to those in [6, 17] to optimize our rationality result. We will use the notation from Section 4 and the setup in Lemma 4.15. We further assume the graph Γ to be *finite* for our discussion until Theorem 5.10, where we use restriction of domain (Proposition 3.9).

Given the choices of the finite index subgroups $W(a_v)$ of W_v satisfying (4–1) for all arcs $a_v \in A_v$ for any vertex v, let $D_v := \cap_{a_v \in A_v} W(a_v)$ be the intersection, which is also finite index in W_v. For each edge e of Γ, let $D_e := o_e^{-1} D_{o(e)} \cap t_e^{-1} D_{t(e)}$ and $W_e := G_e/D_e$. Then W_e is a finite abelian group with induced homomorphisms $\overline{o_e} : W_e \to W_{o(e)}/D_{o(e)}$ and $\overline{t_e} : W_e \to W_{t(e)}/D_{t(e)}$.

We are going to consider all abstract pieces that potentially appear as a component of S_v for some simple relative admissible surface S for some rational chain supported on finitely many hyperbolic elements $\underline{g} = \{g_i : i \in I\}$ represented by tight loops $\underline{\gamma} = \{\gamma_i : i \in I\}$. Thus we extend some previous definitions as follows.

Definition 5.1 For each vertex v, *a piece C at v* is a surface with corners in the thickened vertex space $N(X_v)$ satisfying the following properties:

(1) C has a unique polygonal boundary where edges alternate between arcs in A_v and turns connecting them, where a turn (with the induced orientation) going from a_v to a'_v is an arc supported on X_e connecting the terminus point of a_v and initial point of a'_v.

(2) C is either a disk or an annulus depending on its winding number $w(C)$ as follows, where the winding number is the conjugacy class in G_v represented by the polygonal boundary of C.

  (a) C is a disk if $w(C)$ is trivial in G_v. The map in the interior of C is a null-homotopy of its polygonal boundary in $N(X_v)$.

  (b) C is an annulus if its winding number $w(C)$ is non-trivial, where the other boundary has no corners and represents a loop in X_v homotopic to the polygonal boundary in $N(X_v)$, and the homotopy gives the interior of C.

As before, we define a piece C at v to be *disk-like* if there is an arc $a_v \in A_v$ on the polygonal boundary of C such that $w(C) \in W(a_v)$.

For each vertex v of Γ, let T_v be the set of triples $(a_v, \bar{w}, a'_v)$ where $a_v, a'_v \in A_v$ are arcs with terminus point of a_v and initial point of a'_v on a common edge space X_e, and $\bar{w} \in W_e$. Note that T_v is a *finite* set since there are finitely many pairs of (a_v, a'_v) as above and $\bar{w}$ lies in a finite group given each such a pair. We use such a triple to record a turn from a_v to a'_v with winding number in the coset $\bar{w}$.

Consider a piece C at vertex v with m turns. For each turn, say supported on X_e and going from a_v to a'_v with winding number $w \in G_e$, the triple $(a_v, \bar{w}, a'_v)$ is in T_v, where $\bar{w}$ is the image of w in the quotient W_e. Regard this triple as a basis vector in $\mathbb{R}^{T_v}$ and let $x(C) \in \mathbb{R}^{T_v}$ be the sum of such triples over the m turns of C.

Let $\partial : \mathbb{R}^{T_v} \to \mathbb{R}^{A_v}$ be the rational linear map given by $\partial(a_v, \bar{w}, a'_v) := a'_v - a_v$ for all $(a_v, \bar{w}, a'_v) \in T_v$. Then $\partial(x(C)) = 0$ as the polygonal boundary of C closes up.

There is a pairing on the set $\cup_v T_v$ similar to the pairing of turns (Figure 3). For each triple $(a_v, \bar{w}, a'_v) \in T_v$ with $\bar{w} \in W_e$, there is a unique triple $(a_u, -\bar{w}, a'_u) \in T_u$ such that a_v and a_u are followed by a'_u and a'_v in $\underline{\gamma}$ respectively, where u is necessarily the vertex adjacent to v via e. We say such two triples are *paired*.

Denote the convex rational polyhedral cone $\mathbb{R}^{T_v}_{\geq 0} \cap \ker \partial$ by $\mathcal{C}_v$. For any simple relative admissible surface S for any rational chain supported on $\underline{\gamma}$, let $x(S_v) = \sum x(C)$, where the sum is taken over all pieces C of S_v. Then $x(S_v)$ is an integer point in $\mathcal{C}_v$ and paired turns are encoded by paired triples. Let $x(S) \in \prod_v \mathcal{C}_v$ be the element with v-coordinate $x(S_v)$.

Let $\mathcal{C}(\underline{\gamma})$ be the subspace of $\prod_v \mathcal{C}_v$ consisting of points satisfying the *gluing condition* which we now describe. This is the analog of (3–1) on $\prod_v \mathcal{C}_v$. For each triple $(a_v, \bar{w}, a'_v) \in T_v$, let $\#_{(a_v, \bar{w}, a'_v)}$ be the linear function taking the $(a_v, \bar{w}, a'_v)$ coordinate. We say $x \in \mathcal{C}(\underline{\gamma})$ satisfies the gluing condition if $\#_{(a_v, \bar{w}, a'_v)}(x) = \#_{(a_u, -\bar{w}, a'_u)}(x)$ for all paired triples $(a_v, \bar{w}, a'_v)$ and $(a_u, -\bar{w}, a'_u)$. It follows that $\mathcal{C}(\underline{\gamma})$ is a rational polyhedral cone.

For each $\gamma_i \in \underline{\gamma}$, fix an arc a_i in some A_v supported on γ_i and let $\#_{\gamma_i} = \sum \#_{(a_i, w, a'_v)}$, where the sum is taken over all triples in T_v starting with a_i. For any $x \in \mathcal{C}(\underline{\gamma})$, the gluing condition implies $\#_{\gamma_i}(x)$ is independent of the choice of arc a_i on γ_i. Roughly speaking, the rational linear function $\#_{\gamma_i}(x)$ counts how many times x winds around γ_i.

For $\boldsymbol{r} = (r_i) \in \mathbb{Q}^I_{\geq 0}$ (I is the index set of $\underline{\gamma}$), let $c(\boldsymbol{r})$ be the rational chain $\sum r_i g_i$, and let $\mathcal{C}(\boldsymbol{r})$ be the set of $x \in \mathcal{C}(\underline{\gamma})$ satisfying the *normalizing condition* $\#_{\gamma_i}(x) = r_i$ for all $i \in I$. Let $h(\boldsymbol{r})$ be the homology class of $c(\boldsymbol{r})$ under the projection $H_1(G; \mathbb{R}) \to H_1(\Gamma; \mathbb{R})$. Then h is a rational linear map.

Lemma 5.2 *The set $\mathcal{C}(\boldsymbol{r})$ is nonempty if and only if $\boldsymbol{r} \in \ker h$, in which case, it is a compact rational polyhedron depending piecewise linearly on $\boldsymbol{r}$.*

Proof The chain $c(\mathbf{r})$ bounds an admissible surface relative to vertex groups if and only if $h(\mathbf{r}) = 0$, according to the computation (2–1). Whenever a relative admissible surface exists, it can be simplified to one in simple normal form, which is encoded as a vector in $\mathcal{C}(\mathbf{r})$. Then in this case, each $\mathcal{C}(\mathbf{r})$ is the intersection of a rational polyhedral cone $\mathcal{C}(\gamma)$ with a rational linear subspace $\cap_i \{x : \#_{\gamma_i}(x) = r_i\}$ that depends linearly on $\mathbf{r}$. Thus $\overline{\mathcal{C}}(\mathbf{r})$ is a closed rational polyhedron depending piecewise linearly on $\mathbf{r}$. It is compact since the equations $\#_{\gamma_i}(x) = r_i$ impose upper bounds on all coordinates. □

Definition 5.3 For each vertex v, an integer point $d \in \mathcal{C}_v$ is a *disk-like vector* if $d = x(C)$ for some disk-like piece C at v. Let $\mathcal{D}(v)$ be the set of disk-like vectors.

For any $x \in \mathcal{C}_v$, we say $x = x' + \sum t_j d_j$ is an *admissible expression* if $x' \in \mathcal{C}_v$, $d_j \in \mathcal{D}(v)$ and $t_j \geq 0$. Define

$$\kappa_v(x) := \sup \left\{ \sum t_j \mid x = x' + \sum t_j d_j \text{ is an admissible expression} \right\}.$$

The key point of our encoding and the definitions of D_v, D_e, W_e is to have enough information on the winding numbers of turns to tell whether a piece is disk-like.

Lemma 5.4 *For each $\bar{w} \in W_e$, fix an arbitrary lift $w \in G_e$. Then any disk-like vector $d \in \mathcal{C}_v$ can be realized as a disk-like piece C at v such that every turn from a_v to a'_v on the polygonal boundary of C representing a triple $(a_v, \bar{w}, a'_v)$ has winding number w.*

Proof By definition, there is some disk-like piece C_0 at v realizing the given disk-like vector d. Thus, for some $a_{v,0} \in A_v$, the winding number $w(C_0)$ lies in $W(a_{v,0}) \subset W_v$, which is central in G_v. Suppose C_0 contains a turn from a_v to a'_v supported on the edge space X_e representing a triple $(a_v, \bar{w}, a'_v)$ with actual winding number $w_0 \in G_e$ and $e = e_{out}(a_v)$. Then $w - w_0 \in D_e$ and changing the winding number of the turn from w_0 to w would change the winding number $w(C_0)$ of the piece C_0 by $o_e(w - w_0) \in D_v$. Since $D_v \subset W(a_{v,0})$, such a change preserves the property of being disk-like. After finitely many such changes, we modify C_0 to a disk-like piece C with the desired winding numbers of turns. □

Any admissible surface S naturally provides an admissible expression for $x(S_v)$ by sorting out disk-like pieces among components of S_v. Hence $\kappa_v(x(S_v))$ is no less than the number of disk-like pieces in S_v.

For each vector $x \in \mathcal{C}_v \subset \mathbb{R}_{\geq 0}^{T_v}$, denote its ℓ^1-norm by $|x|$. Then $|\cdot|$ on $\mathcal{C}_v$ coincides with the linear function taking value 1 on each basis vector. Thus $\sum_v |x(S_v)|$ is the total number of turns in S, which is twice the number of corners.

Recall from Definition 4.14 that $\widehat{\chi}(S)$ is the over-counting Euler characteristic that counts disk-like pieces as disks in a simple relative admissible surface S.

Lemma 5.5 *Fix any $\boldsymbol{r}$ as above.*

(1) *For any S simple relative admissible for $c(\boldsymbol{r})$ of degree n, we have $x(S)/n \in \mathcal{C}(\boldsymbol{r})$ and*

$$\frac{-\widehat{\chi}(S)}{2n(S)} \geq \sum_v \frac{1}{4}|x(S_v)/n| - \sum_v \frac{1}{2}\kappa_v(x(S_v)/n).$$

(2) *For any rational point $x = (x_v) \in \mathcal{C}(\boldsymbol{r})$ and any $\epsilon > 0$, there is a simple relative admissible surface S for $c(\boldsymbol{r})$ of a certain degree n such that $x(S_v)/n = x_v$ and*

$$\frac{-\widehat{\chi}(S)}{2n} \leq \sum_v \frac{1}{4}|x_v| - \sum_v \frac{1}{2}\kappa_v(x_v) + \epsilon.$$

Proof   (1)   It is easy to see $x(S)/n \in \mathcal{C}(\boldsymbol{r})$ from the definition. To obtain the inequality, recall that

$$-\widehat{\chi}(S) = -\sum_C \widehat{\chi}_o(C) = \frac{1}{4}\#\text{corners} - \sum_C \widehat{\chi}(C) = \frac{1}{2}\#\text{turns} - \#\text{disk-like pieces}.$$

Since $\kappa_v(x(S_v))$ is no less than the number of disk-like pieces in S_v, the inequality follows.

(2)   By finiteness of Γ, there are admissible expressions $x_v = x_v' + \sum_j t_{j,v} d_{j,v}$ with each $t_{j,v} \in \mathbb{Q}_{\geq 0}$ such that $\sum_{j,v} t_{j,v} + 2\epsilon > \sum_v \kappa_v(x_v)$. Note that each x_v' is rational as each $t_{j,v}$ and x_v are. Choose an integer n so that each $nt_{j,v}$ is an integer and nx_v' is an integer vector in $\mathcal{C}_v$. For each $\bar{w} \in W_e$, fix a lift $w \in G_e$. We can choose the lifts such that $-w$ is the lift of $-\bar{w}$. By Lemma 5.4, we can realize $nt_{j,v} d_{j,v}$ as the union of $nt_{j,v}$ pieces that are disk-like, such that each turn from a_v to a_v' representing a triple $(a_v, \bar{w}, a_v')$ has winding number w. We can also realize nx_v' as the union of some other pieces at v with turns satisfying the same property. Let S_v be the disjoint union of these pieces at v. Then $x \in \mathcal{C}(\boldsymbol{r})$ and our choice of the lifts imply that the surface $\sqcup_v S_v$ satisfies the gluing condition (3–1) and glues to a simple relative admissible surface S for $c(\boldsymbol{r})$ of degree n with $x(S_v)/n = x_v$. Noticing that the number of disk-like pieces in S is no less than $n\Sigma_{j,v} t_{j,v} > n\Sigma_v \kappa_v(x_v) - 2n\epsilon$, the estimate of $-\widehat{\chi}(S)/2n$ easily follows from a computation similar to the one in the first part.

$\square$

Let $\mathrm{conv}(E)$ denote the convex hull of a set E in some vector space. Denote the Minkowski sum of two sets E and F by $E + F := \{e + f \mid e \in E \text{ and } f \in F\}$. Note that $\mathrm{conv}(E + F) = \mathrm{conv}(E) + \mathrm{conv}(F)$.

The following lemma is the analog of [6, Lemma 3.10] and has the same proof.

Lemma 5.6 (Calegari [6]) *The function κ_v on $\mathcal{C}_v$ is a non-negative concave homogeneous function which takes value 1 exactly on the boundary of $\mathrm{conv}(\mathcal{D}(v)) + \mathcal{C}_v$ in $\mathcal{C}_v$.*

It is an important observation in [17] that both $\mathrm{conv}(\mathcal{D}(v)) + \mathcal{C}_v$ and κ_v are nice, no matter how complicated $\mathcal{D}(v)$ is.

Lemma 5.7 (Chen [17]) *There is a finite subset D' of $\mathcal{D}(v)$ such that $D' + \mathcal{C}_v = \mathcal{D}(v) + \mathcal{C}_v$. Consequently, the function κ_v is the minimum of finitely many rational linear functions.*

Proof The first assertion follows from [17, Lemma 4.7]. Then we have

$$\mathrm{conv}(\mathcal{D}(v)) + \mathcal{C}_v = \mathrm{conv}(\mathcal{D}(v) + \mathcal{C}_v) = \mathrm{conv}(D' + \mathcal{C}_v) = \mathrm{conv}(D') + \mathcal{C}_v.$$

Note that $\mathrm{conv}(D')$ is a compact rational polyhedron since D' is a finite set of integer points. Hence $\mathrm{conv}(D') + \mathcal{C}_v$ is a rational polyhedron as it is the sum of two such polyhedra (see the proof of [1, Theorem 3.5]). Then $\mathrm{conv}(\mathcal{D}(v)) + \mathcal{C}_v = \mathcal{C}_v \cap (\cap_i \{f_i \geq 1\})$ for a finite collection of rational linear functions $\{f_i\}$. Combining with Lemma 5.6, we have $\kappa_v(x) = \min_i f_i(x)$ for all $x \in \mathcal{C}_v$. □

Lemma 5.8 *The optimization $\min \sum_v \frac{1}{4}|x_v| - \sum_v \frac{1}{2}\kappa_v(x_v)$ among $x = (x_v) \in \mathcal{C}(\boldsymbol{r})$ can be computed via linear programming. The minimum is $\mathrm{scl}_{(G,\{G_v\})}(c(\boldsymbol{r}))$, which depends piecewise rationally linearly on $\boldsymbol{r} \in \ker h$ and is achieved at some rational point in $\mathcal{C}(\boldsymbol{r})$.*

Proof Recall that $|x_v|$ is a rational linear function for $x_v \in \mathcal{C}_v$. Combining with Lemma 5.7, for each vertex v, there are finitely many rational linear functions $f_{j,v}$ such that $|x_v|/4 - \kappa_v(x_v)/2 = \max_j f_{j,v}(x_v)$. By introducing slack variables $y = (y_v)$, the optimization is equivalent to minimizing $\sum_v y_v$ subject to $y_v \geq f_{j,v}(x_v)$ for all j, v and $x = (x_v) \in \mathcal{C}(\boldsymbol{r})$, which is a rational linear programming problem in variables (x, y). The minimum depends piecewise rationally linearly on $\boldsymbol{r} \in \ker h$ by Lemma 5.2 and is achieved at a rational point.

The minimum is $\mathrm{scl}_{(G,\{G_v\})}(c(\boldsymbol{r}))$ by Lemma 5.5 and Lemma 4.15. □

Remark 5.9 The function $\sum_v |x_v|$ is actually a constant $\sum_i r_i A_i$ for $x = (x_v) \in \mathcal{C}(\boldsymbol{r})$, where A_i is the number of arcs that the edge spaces cut γ_i into. Thus for a fixed chain, the problem comes down to maximizing the number of disk-like pieces.

Now we return to full generality without assuming Γ to be finite or locally finite.

Theorem 5.10 (Rationality) *Let G be a graph of groups $\mathcal{G}(\Gamma, \{G_v\}, \{G_e\})$ where*

(1) $\mathrm{scl}_{G_v} \equiv 0$, *and*

(2) *the images of edge groups in each vertex group are central and mutually commensurable.*

Then scl_G is piecewise rational linear, and $\mathrm{scl}_G(c)$ can be computed via linear programming for each rational chain $c \in B_1^H(G)$.

Proof We compute $\mathrm{scl}_G(\sum r_i g_i)$ for an arbitrary finite set of element $\underline{g} = \{g_1, \ldots, g_m\} \subset G$ with $r_i \in \mathbb{Q}_{>0}$ so that the chain $\sum r_i g_i$ is null-homologous in G. By Proposition 2.11, we may assume each g_i to be hyperbolic and consider $\mathrm{scl}_{(G, \{G_v\})}(c(\boldsymbol{r}))$ with $\boldsymbol{r} \in \ker h$ instead. By restriction of domain (Proposition 3.9), we further assume Γ to be finite. Then the result follows from Lemma 5.8. □

Remark 5.11 Theorem 5.10 holds with the weaker assumption $\mathrm{scl}_G(c_v) = 0$ for all $c_v \in B_1^H(G_v)$ in place of $\mathrm{scl}_{G_v} \equiv 0$ and all vertices v.

Remark 5.12 Let G be a graph of groups where each edge group is $\mathbb{Z}$ and each vertex group G_v is itself a graph of groups as in Theorem 5.10 with vanishing $H_2(G_v; \mathbb{R})$. Using the method in [9], it follows from Theorem 5.10 that the Gromov–Thurston norm on $H_2(G; \mathbb{R})$ has a rational polyhedral unit ball and can be computed via linear programming.

6   Scl in Baumslag–Solitar groups

6.1   Basic setups

In this section we focus on scl in Baumslag–Solitar groups $G = \mathrm{BS}(M, L) = \langle a, t \mid a^M = t a^L t^{-1} \rangle$ with integers $M, L \neq 0$. We are not interested in the case where $|M| = 1$ or $|L| = 1$ since $\mathrm{BS}(M, L)$ is solvable and $\mathrm{scl}_{\mathrm{BS}(M,L)} \equiv 0$ in such cases. Most of the results and tools are applicable for any graphs of groups with abelian vertex
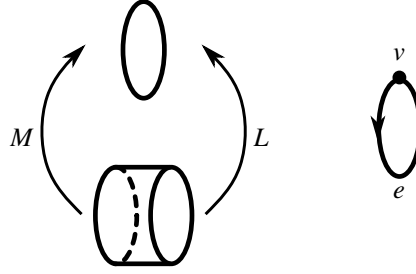
Figure 7: The graph of spaces associated to $\mathrm{BS}(M, L)$ and the underlying graph.

groups, but we will not pursue such generalizations. Let $d := \gcd(|M|, |L|)$, $m := M/d$ and $\ell := L/d$. Denote by $h : G \to \mathbb{Z}$ the homomorphism given by $h(a) = 0$ and $h(t) = 1$. An element g is *t-balanced* if $h(g) = 0$.

We denote the only vertex and edge by v and $\{e, \bar{e}\}$ respectively, where e is oriented to represent the generator t. See Figure 7. Notation from Section 5 will be used.

Let $g = a^{p_1} t^{\epsilon_1} \ldots a^{p_n} t^{\epsilon_n}$ be a cyclically reduced word, where $\epsilon_i = \pm 1$ for all i. Then g is represented by a tight loop γ in X_G cut into n arcs $A_v = \{a_i \mid 1 \leq i \leq n\}$ where a_i has winding number $w(a_i) = p_i$. Note that we have the transition map $\tau_e : |M|\mathbb{Z} \to |L|\mathbb{Z}$ with $\tau_e(x) = -Lx/M$. For each $1 \leq i \leq n$, let $\mu_i := \max_{0 \leq k \leq n} \sum_{j=1}^{k} \epsilon_{i+j}$ and $\lambda_i := -\min_{0 \leq k \leq n} \sum_{j=1}^{k} \epsilon_{i+j}$, where indices are taken mod n and the summation is 0 when $k = 0$. It is straightforward to see that $\mathrm{Dom}\tau_{P(a_i)} = dm^{\mu_i} \ell^{\lambda_i} \mathbb{Z}$, $\mathrm{Im}\tau_{P(a_i)} = dm^{\mu_i - h(g)} \ell^{\lambda_i + h(g)} \mathbb{Z}$, and $\tau_{P(a_i)}(x) = (-1)^{h(g)} \ell^{h(g)} x / m^{h(g)}$. Example 4.4 (with $X = dm^{\mu_i} \ell^{\lambda_i}$ and $Y = \pm dm^{\mu_i - h(g)} \ell^{\lambda_i + h(g)}$) shows that, letting

$$(6\text{–}1) \qquad W_0(a_i) := \begin{cases} dm^{\mu_i - |h(g)|} \ell^{\lambda_i} \mathbb{Z} & \text{if } h(g) \geq 0, \\ dm^{\mu_i} \ell^{\lambda_i - |h(g)|} \mathbb{Z} & \text{if } h(g) \leq 0, \end{cases}$$

we have $W_0(a_i) \subset \mathrm{Dom}\tau_{P(a_i)}^{p} + \mathrm{Im}\tau_{P(a_i)}^{q}$ for all $p, q \geq 0$.

In the sequel, we will use the following two different setups.

(1) **Setup 1**: Let $W(a_i) = W_0(a_i)$ for all i.

(2) **Setup 2**: Let $W(a_i) = \cap_j W_0(a_j)$ for all i.

In both setups, the group D_v defined as $\cap_i W(a_i)$ in Section 5 equals $\cap_i W_0(a_i)$, and thus the choice of setups does not affect our encoding or the space $\mathcal{C}_v$. The only difference is that we have more disk-like pieces or vectors in Setup 1 than Setup 2. We will explicitly state our choice of setups whenever the discussion depends on it.

For an explicit formula of D_v, let $\lambda := \max_i \lambda_i$ and $\mu := \max_i \mu_i$. Since $h(g) = \sum \epsilon_i$, it is easy to observe that $\mu - |h(g)| = \lambda \geq 0$ when $h(g) \geq 0$ and $\lambda - |h(g)| = \mu \geq 0$ when $h(g) \leq 0$. In any case, using the formula (6–1) we have

$$D_v = dm^{\rho(g)} \ell^{\rho(g)} \mathbb{Z},$$

where $\rho(g) := \min(\mu, \lambda)$, which we call the *complexity* of g.

When $h(g) = 0$, this can be easily seen geometrically. The infinite cyclic cover $\widetilde{X}_G$ of X_G corresponding to $\ker h$ has a $\mathbb{Z}$-action by translation with fundamental domains projecting homeomorphically to the thickened vertex space $N(X_v)$. Since $h(g) = 0$, the tight loop γ representing g lifts to a loop $\tilde{\gamma}$ on $\widetilde{X}_G$, and $\rho(g) + 1$ is the number of fundamental domains that $\tilde{\gamma}$ intersects. In particular, when $h(g) = 0$, the element g is t-alternating if and only if the complexity $\rho(g) = 1$.

More generally, for a chain $c = \sum r_i g_i$ with each $r_i \neq 0$, define its complexity $\rho(c) := \max_i \rho(g_i)$. Then

(6–2) $$D_v = dm^{\rho(c)} \ell^{\rho(c)} \mathbb{Z},$$

and $\rho(c)$ controls the amount of information we need to encode. Denote $|D_v| := d|m|^{\rho(c)}|\ell|^{\rho(c)}$.

Using the notation from Section 5, we have $D_e = m^{\rho(c)} \ell^{\rho(c)} \mathbb{Z}$ and $W_e = \mathbb{Z}/D_e$ in both setups. The fact that D_e does not depend on d is important in Theorem 6.29.

To better understand integer points in $\mathcal{C}_v$ and disk-like vectors, consider a directed graph Y with vertex set A_v, where each oriented edge from a_i to a_j corresponds to a triple $(a_i, \bar{w}, a_j) \in T_v$. See Proposition 6.28 and Figure 16 for an example. Then each vector $x \in \mathcal{C}_v$ assigns non-negative weights to edges in Y.

Define the support $\text{supp}(x)$ to be the subgraph containing edges with positive weights. Then $\text{supp}(x)$ is a union of positively oriented cycles in Y. Note that an integer point $x \in \mathcal{C}_v$ can be written as $x(C)$ for some piece C if and only if $\text{supp}(x)$ is connected. If two pieces C and C' are encoded by the same vector x, then their winding numbers $w(C)$ and $w(C')$ are congruent mod D_v since the vertex group is abelian. Moreover, fixing a lift $\widetilde{w} \in \mathbb{Z}$ of each $\bar{w} \in W_e$, we can compute this winding number $w(x)$ as a linear function on $\mathbb{R}^{T_v}$ determined by

(6–3) $$w(a_i, \bar{w}, a_j) := w(a_j) + \iota(\widetilde{w})$$

where $\iota = t_e$ if $e_{out}(a_i) = \bar{e}$ (ie a_i leaves v by following $\bar{e}$, see Figure 3) and $\iota = o_e$ if $e_{out}(a_i) = e$. Then $w(x)$ depends on the choice of lifts but $w(x)$ mod D_v does not. In particular, it makes sense to discuss whether $w(x) \in W(a_i)$ since $D_v \subset W(a_i)$.

Then in both setups, an integer point $x \in C_v$ is disk-like if and only if supp(x) is connected and $w(x) \in W(a_i)$ for some a_i in supp(x). Since $W(a_i) = D_v$ for all i in Setup 2, the criterion is simply that supp(x) is connected and $w(x) \in D_v$. In contrast, in Setup 1, we have more disk-like pieces, which makes it easier to construct simple relative admissible surfaces using disk-like pieces.

We use $\mathcal{C}(c)$ instead of $\mathcal{C}(\boldsymbol{r})$ to denote the polyhedron encoding normalized simple relative admissible surfaces for c, as we will not consider families of chains with varying $\boldsymbol{r}$.

6.2 Extremal surfaces

The goal of this subsection is to obtain a criterion for the existence of extremal surfaces in the case of Baumslag–Solitar groups. We accomplish this by strengthening results in Section 4 to analyze when the asymptotic approximation can terminate at a finite stage. We will use the notion of transition maps and adjustment maps from Section 4.

Let $c = \sum r_i g_i$ be a rational chain as in the previous subsection where $r_i \in \mathbb{Q}_{>0}$ and each $g_i \in \underline{g}$ is a hyperbolic element represented by a tight loop $\gamma_i \in \underline{\gamma}$. Throughout this subsection, we use Setup 2 where $W(a) = D_v$ for all $a \in A_v$.

Lemma 6.1 *Let S be a simple relative admissible surface of degree n with*

$$-\frac{\widehat{\chi}(S)}{2n} = \mathrm{scl}_{\mathrm{BS}(M,L)}(c).$$

Then S consists of disk-like pieces only.

Proof Suppose there is a piece C not disk-like. Take a piece $\widetilde{C}$ where the boundary is a degree $|D_v|$ cover of ∂C. Then $\widetilde{C}$ is disk-like. Moreover, $\widetilde{C}$ and $|D_v|$ copies of every piece in S other than C together satisfy the gluing condition and form a new admissible surface $\widetilde{S}$ of degree $n|D_v|$, such that

$$-\frac{\widehat{\chi}(\widetilde{S})}{2n|D_v|} = -\frac{\widehat{\chi}(S)}{2n} - \frac{1}{2n|D_v|} < \mathrm{scl}_{\mathrm{BS}(M,L)}(c).$$

This is absurd (in view of Lemma 4.15) since S is optimal. $\qquad\square$

The optimal solution to the linear programming in Lemma 5.8 to compute $\mathrm{scl}_{\mathrm{BS}(M,L)}(c)$ provides a surface S satisfying the assumption in Lemma 6.1. To obtain an extremal surface we need to adjust disk-like pieces into genuine disks by trivializing their winding numbers.

This can be easily understood when $M = L$. The winding number $w(C)$ is divisible by $|D_v| = |M| = |L|$ since C is disk-like, which can be eliminated and added to $w(C')$ by an adjustment where C' is any nearby piece. Eventually we can concentrate all the winding numbers at a single piece (assuming S is connected). The last winding number is simply the sum of all original winding numbers which remains invariant in the adjustment process. Thus we can make S into an extremal surface if and only if this sum vanishes. A similar argument works when $M = -L$ where one needs to consider an "alternating" sum instead.

In what follows we assume $M \neq \pm L$, which is the hard case. If we were allowed to treat winding numbers as rational numbers, the same process as above can be done, and the problem comes down to the vanishing of an obstruction number. The main bulk of this subsection is to use suitable finite covers and stability (Lemma 4.3) to make our situation as good as working over $\mathbb{Q}$.

In the sequel, we always suppose S to be a simple relative admissible surface consisting of disk-like pieces only. Recall that there is a finite graph Γ_S encoding how pieces glue up to form S, which has a graph homomorphism π to the underlying graph Γ. In the case of Baumslag–Solitar groups, Γ is a circle with one edge e (shown in Figure 7), inducing an orientation on Γ_S. Recall from Section 4 that every nontrivial oriented path P in Γ_S has an adjustment map $\alpha(P)$, which is of the form $\alpha(P)(x) = (-\ell/m)^{h(P)}x$ for every $x \in \mathrm{Dom}\alpha(P) \subset d\mathbb{Z}$, where

$$h(P) = \#(\text{positively oriented edges on P}) - \#(\text{negatively oriented edges on P}).$$

Similarly, for each oriented cycle ω in Γ_S, define $h(\omega)$ to be $h(P)$, where we consider ω as an oriented path P by choosing an arbitrary base point.

Definition 6.2 An oriented cycle ω in Γ_S is *imbalanced* if the number $h(\omega)$ above is non-zero (so that $(-\ell/m)^{h(\omega)} \neq 1$ since $M \neq \pm L$). Otherwise, we say ω is *balanced*.

We investigate whether all disk-like pieces can be made into genuine disks in each component Σ of S. Denote the corresponding component of Γ_S by Γ_Σ. We say Σ is *disk-only* if it consists of genuine disks.

The following lemmas show that imbalanced cycles are useful to eliminate winding numbers of pieces.

Lemma 6.3 *Suppose Γ_Σ contains an imbalanced oriented cycle ω, on which sits a piece C as the base point. If $w(C) \in \mathrm{Dom}\alpha(\omega)^p + \mathrm{Im}\alpha(\omega)^q$ for all $p, q \geq 0$, then for any $N \in \mathbb{Z}_+$, there is an adjustment of Σ supported on C, after which $w(C)$ is divisible by $dm^N \ell^N$.*

Proof Without loss of generality, assume $h(\omega) > 0$. Then $\mathrm{Dom}\alpha(\omega)^p \subset dm^p\mathbb{Z}$ and $\mathrm{Im}\alpha(\omega)^q \subset d\ell^q\mathbb{Z}$. By the assumption, there are $a, b \in \mathbb{Z}$ and $u \in \mathrm{Dom}\alpha(\omega)^{2N}$, $v \in \mathrm{Im}\alpha(\omega)^{2N}$, such that $w(C) = au + bv$. Then $\alpha(\omega)^N(au) \in \mathrm{Dom}\alpha(\omega)^N \cap \mathrm{Im}\alpha(\omega)^N$ is divisible by $dm^N\ell^N$, so is $\alpha(\omega)^{-N}(bv)$. $\square$

Lemma 6.4 *Suppose Γ_Σ contains a balanced oriented cycle ω, on which sits a piece C as the base point. Let $\widetilde{S}$ be a finite cover of S and $\widetilde{\omega}$ be a degree k lift of ω, where the preimages of C are denoted as $\widetilde{C}_1, \ldots, \widetilde{C}_k$. If $w(C) \in \mathrm{Dom}\alpha(\omega)^p + \mathrm{Im}\alpha(\omega)^q$ for all $p, q \geq 0$, then there is an adjustment of $\widetilde{\Sigma}$ supported on $\{\widetilde{C}_1, \ldots, \widetilde{C}_k\}$, after which $w(\widetilde{C}_1) = kw(C)$ and $w(\widetilde{C}_j) = 0$ for all $k > 1$.*

Proof We eliminate $w(\widetilde{C}_j)$ for $j > 1$ as in the proof of Lemma 4.10. The fact that ω is balanced implies $w(\widetilde{C}_1) = kw(C)$ after the adjustment. $\square$

Lemma 6.5 *Suppose a component Σ contains two distinct embedded cycles ω and ω_{im} where ω_{im} is imbalanced. If all pieces of Σ are genuine disks except for one piece C, then there is a finite cover of Σ where all pieces are genuine disks after a suitable adjustment, assuming $w(C)$ is divisible by $dm^N\ell^N$ for some $N \in \mathbb{Z}_+$ depending on Σ.*

Proof It suffices to prove the case where C lies on the cycle ω. For the general case, we can increase N by the diameter of Γ_Σ, and move $w(C)$ to a piece C' on ω by an adjustment supported on the two pieces, after which $w(C')$ is divisible by $dm^N\ell^N$ and $w(C) = 0$. When C lies on ω, fix an embedded oriented path P from C to ω_{im}. Since ω and ω_{im} are embedded and distinct, for any $k \in \mathbb{Z}_+$, there is a degree k normal cover $\widetilde{\Sigma}$ of Σ containing cycles $\widetilde{\omega}_{im,j}$ and $\widetilde{\omega}$, $1 \leq j \leq k$, where $\widetilde{\omega}$ covers ω with degree k and each $\widetilde{\omega}_{im,j}$ projects homeomorphically to ω_{im}. See Figure 8. Denote the lifts of C according to the cyclic order on $\widetilde{\omega}$ by $\widetilde{C}_j$, $1 \leq j \leq k$. Let $\widetilde{P}_j$ be the lift of P connecting $\widetilde{C}_j$ and $\widetilde{\omega}_{im,j}$.

Fix an integer $k_0 \in \mathrm{Dom}\alpha(P\omega_{im}\overline{P})$ of the form $dm^{N_0}\ell^{N_0}$ with $N_0 \in \mathbb{Z}_+$, where $\overline{P}$ is P reversed and $P\omega_{im}\overline{P}$ is the concatenation. Note that any $\widetilde{C}_j$ with $w(\widetilde{C}_j)$ divisible by $w_0 := [(-\ell/m)^{h(\omega_{im})} - 1]k_0 \neq 0$ can be made into a genuine disk without affecting any other pieces on $\widetilde{\Sigma}$ since

$$w_0 = [(-\ell/m)^{h(\omega_{im})} - 1]k_0 = \alpha(P\omega_{im}\overline{P})k_0 - k_0 = \alpha(\widetilde{P}_j\widetilde{\omega}_{im,j}\overline{\widetilde{P}}_j)k_0 - k_0.$$

Our strategy is to choose a suitable covering degree k and make adjustment so that each $w(\widetilde{C}_j)$ is divisible by w_0.
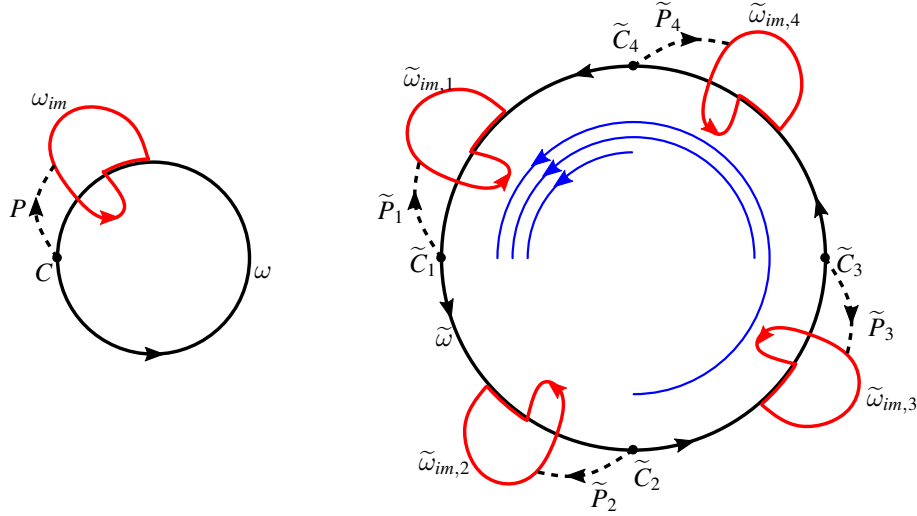
Figure 8: The cover unwrapping ω when $k = 4$ with the blue concentric arcs indicating our adjustment moving winding numbers of $\widetilde{C}_j$ together in the case where ω is balanced

If ω is balanced, let $k = |w_0|$. By stability (Lemma 4.3), we can apply the adjustment in Lemma 6.4 by choosing N large. After the adjustment, the only non-trivial winding number in $\widetilde{\Sigma}$ is $w(\widetilde{C}_1) = kw(C)$, which is divisible by w_0 and thus can be eliminated.

If ω is imbalanced, let $k = |h(\omega_{im})w_0|$ and $s = \text{sign}(h(\omega))$. We divide $\{\widetilde{C}_1, \ldots, \widetilde{C}_k\}$ into $|h(\omega_{im})|$ groups to carry out the adjustment. For each $1 \leq i \leq |h(\omega_{im})|$, there is an adjustment supported on $\{\widetilde{C}_{i+jsh(\omega_{im})} \mid 1 \leq j \leq |w_0|\}$ (subscripts are taken mod k) along subpaths of $\widetilde{\omega}$, such that $w(\widetilde{C}_{i+jsh(\omega_{im})}) = 0$ for all j except

$$(6\text{–}4) \quad w(\widetilde{C}_i) = \sum_{j=0}^{|w_0|-1} \alpha(\omega)^{sjh(\omega_{im})} w(C) = |w_0|w(C) + \sum_{j=0}^{|w_0|-1} [\alpha(\omega)^{sjh(\omega_{im})} w(C) - w(C)].$$

This can be done provided $w(C) \in \text{Dom}\,\alpha(\omega)^{sjh(\omega_{im})}$ for all $1 \leq j \leq |w_0|$ by choosing N large. Make N further larger so that $w(C) \in m^{N_0}\ell^{N_0}\text{Dom}\,\alpha(\omega)^{sjh(\omega_{im})}$ for all $1 \leq j \leq |w_0|$. Then $(x - 1)^\lambda w(C)$ is divisible by $w_0 = (x - 1)dm^{N_0}\ell^{N_0}$ for all $1 \leq \lambda \leq |w_0 h(\omega)|$, where $x = (-\ell/m)^{h(\omega_{im})}$. Therefore, for any $j \geq 0$,

$$\alpha(\omega)^{sjh(\omega_{im})} w(C) - w(C) = [x^{j|h(\omega)|} - 1]w(C) = \sum_{\lambda=1}^{j|h(w)|} \binom{j|h(\omega)|}{\lambda}(x - 1)^\lambda w(C)$$

is an integer multiple of w_0 since each term in the summation is. Thus for N large, each term in the last summation of equation (6–4) is divisible by w_0. Hence $w(\widetilde{C}_i)$ is also divisible by w_0 and can be eliminated. $\qquad\square$
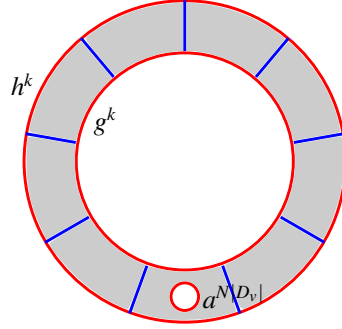
Figure 9: Pseudo-inverses g and h have powers almost cobounding an annulus.

Definition 6.6 Two hyperbolic elements g and h are *pseudo-inverses* if $\text{scl}_{\text{BS}(M,L)}(g + h) = 0$. A rational chain $c = \sum r_i g_i$ with $r_i \in \mathbb{Q}_{>0}$ and each $g_i \in \underline{g}$ hyperbolic is *reduced* if no g_i^p and g_j^q are pseudo-inverses for any $p, q \in \mathbb{Z}_+$ and any $g_i, g_j \in \underline{g}$ (possibly $i = j$).

Lemma 6.7 *Let $g, h \in \text{BS}(M, L)$ be hyperbolic elements with positive* scl. *Let $\rho = \rho(g + h)$ be the complexity and $|D_v| = d|m|^\rho|\ell|^\rho$ as in (6–2). Then the following are equivalent:*

(1) *g and h are pseudo-inverses;*

(2) *For some $k \in \mathbb{Z}_+$ and $N \in \mathbb{Z}$, there is a thrice punctured sphere S in $\text{BS}(M, L)$ in simple normal form with Γ_S being a cycle such that S bounds conjugacy classes g^k, h^k and $a^{N|D_v|}$ (see Figure 9);*

(3) *For some $k \in \mathbb{Z}_+$ and $N \in \mathbb{Z}$, some cyclic conjugates of g^k and h^k can be written as cyclically reduced words*

$$a^{n_1} t^{\epsilon_1} \dots a^{n_s} t^{\epsilon_s}, \quad \text{and} \quad t^{-\epsilon_s} a^{-n_s} \dots a^{-n_2} t^{\epsilon_1} a^{-n_1 + N|D_v|},$$

respectively, where all $\epsilon_i = \pm 1$.

Proof Obviously (2) and (3) are equivalent. Now suppose (2) holds. Notice that the only non-disk piece in the thrice punctured sphere S is disk-like. Thus $\widehat{\chi}(S) = 0$ and hence $\text{scl}(g + h) = \text{scl}(g^k + h^k)/k = 0$ by Lemma 4.15. Therefore (1) holds.

Conversely, suppose g and h are pseudo-inverses. By Theorem 5.10, the linear programming produces a simple relative admissible surface S with $\widehat{\chi}(S) = 0$. By Lemma 6.1, all pieces of S are disk-like, thus $\chi(\Gamma_S) = \widehat{\chi}(S) = 0$. Since each vertex of

Γ_S has valence at least 2 (Lemma 4.11), it follows that each component of Γ_S is a cycle. Since both g and h have positive scl and $\mathrm{scl}(g + h) = 0$, each component must bound $g^k + h^k$ for some $k \in \mathbb{Z}$ relative to the vertex group. By Lemma 6.3 and Lemma 6.4, after taking a suitable finite cover of a component and an adjustment, all but at most one disk-like piece become genuine disks, as depicted in Figure 9. Thus (2) follows. $\square$

The last part of the proof without much change implies the following characterization of chains with trivial scl in $\mathrm{BS}(M, L)$.

Proposition 6.8 *Let $c = \sum r_i g_i$ be a rational chain with $r_i \in \mathbb{Q}_{>0}$ and each $g_i \in \underline{g}$ hyperbolic. Then $\mathrm{scl}_{\mathrm{BS}(M,L)}(c) = 0$ if and only if c can be rewritten as $\sum r'_j(h_j + h'_j)$ where $r'_j \in \mathbb{Q}_{>0}$, h_j and h'_j are pseudo-inverses and each of the form g_i^k for some $g_i \in \underline{g}$ and $k \in \mathbb{Z}_+$. In particular, reduced chains have positive scl.*

For simplicity, we only consider reduced chains in the rest of this subsection.

We first characterize the existence of imbalanced cycles. In the sequel, let $\widetilde{\Gamma}$ be the universal cover of Γ, considered as a bi-infinite line with vertex set $\mathbb{Z}$ and oriented edges $k \to k + 1$ for each $k \in \mathbb{Z}$.

Lemma 6.9 *For a component Σ of S, the following are equivalent:*

(1) *Γ_Σ contains an imbalanced cycle;*

(2) *Γ_Σ contains an embedded imbalanced cycle;*

(3) *There is no graph homomorphism $\phi : \Gamma_\Sigma \to \widetilde{\Gamma}$ preserving edge orientations.*

Proof The existence of such a map ϕ implies that each cycle is balanced. Breaking an imbalanced cycle into embedded ones, we obtain at least one embedded imbalanced cycle. Finally, we can construct ϕ inductively by first mapping a chosen vertex to an arbitrary vertex of $\widetilde{\Gamma}$ and then extending the map to adjacent vertices preserving edge orientations. Such a construction ends up with a well-defined ϕ if all embedded cycles in Γ_Σ are balanced. $\square$

We show a component Σ of S can be made disk-only after taking a finite cover if it contains an imbalanced cycle ω. This is achieved by taking suitable covers and adjustment so that we land in the situation of Lemma 6.5. We need the following lemma on finite covers of graphs.

Lemma 6.10 *Let $\omega_1, \ldots, \omega_n$ be non-trivial cycles in a connected graph Λ. Then for any $N \in \mathbb{Z}_+$, there is a connected finite normal cover $\widetilde{\Lambda} \to \Lambda$ such that each lift $\widetilde{\omega}_i \to \omega_i$ has degree divisible by N, for any $1 \leq i \leq n$.*

Proof Let $w_1 = x_{i_k}^{e_k} \ldots x_{i_1}^{e_1}$ be a cyclically reduced word representing the conjugacy class of ω_1 in the free group $\pi_1(\Lambda)$ with generators $\{x_1, \ldots x_r\}$, where each $e_j = \pm 1$. Construct permutations $\varphi(x_1), \ldots, \varphi(x_r)$ on the set $\mathcal{E} = \{1, \ldots, Nk\}$ such that

$$\varphi(x_{i_j})(qk + j) = \begin{cases} qk + j + 1 & \text{if } e_j = 1 \\ qk + j - 1 & \text{if } e_j = -1 \end{cases},$$

for $q = 0, \ldots, N - 1$ and $j = 1, \ldots, k$, where elements in $\mathcal{E}$ are represented mod Nk. For each generator x_i, the conditions imposed on each permutation $\varphi(x_i)$ above are compatible (well-defined and injective). Thus such permutations exist and induce an action φ of $\pi_1(\Lambda)$ on $\mathcal{E}$ by permutations. By construction, we have $\varphi(w_1)(qk + 1) = (q + 1)k + 1$ for all $q = 0, \ldots, N - 1$, which gives an orbit of length N. Let N_1 be the kernel of φ, and then w_1 has order divisible by N in the quotient $\pi_1(\Lambda)/N_1$. Similarly construct normal subgroups N_i for each cycle ω_i. Then the normal cover corresponding to $\cap_i N_i$ has the desired property. □

Recall that each boundary component B of S provides an oriented non-backtracking (by Lemma 4.11) cycle $\omega_B \subset \Gamma_S$ with $h(\omega_B) = kh(g_i)$ if B wraps k times around γ_i. In particular, ω_B is balanced if and only if g_i is t-balanced. Since Setup 2 is used, for any disk-like piece C on ω_B, we have $w(C) \in \text{Dom}\alpha(\omega_B)^p + \text{Im}\alpha(\omega_B)^q$ for all $p, q \geq 0$.

Lemma 6.11 *If a component Σ of S contains an imbalanced cycle ω, then for any $N \in \mathbb{Z}_+$, there is an adjustment on a connected finite cover $\widetilde{\Sigma}$ of Σ, after which all pieces of $\widetilde{\Sigma}$ have winding number 0, except for one piece $\widetilde{C}$ which has $w(\widetilde{C})$ divisible by $dm^N \ell^N$ and sits on a lift of ω.*

Proof Fix a piece C on ω. For any other piece C', choose a path $P(C')$ on Γ_Σ from C' to C. Then there is some $N(C') \in \mathbb{Z}_+$ such that $n(C') = d|m|^{N(C')}|\ell|^{N(C')}$ is in the domain of $\alpha(P(C'))$ and its image lies in D_v. Choose also a boundary component $B(C')$ passing through C'.

Now for any C' with $\omega_{B(C')}$ imbalanced, by stability and Lemma 6.3, we may assume $w(C')$ to be divisible by $n(C')$. Consider the finite collection of all (non-backtracking) cycles $\omega_{B(C')}$ that are *balanced* with corresponding integers $n(C')$. Applying Lemma 6.10, we obtain a cover $\widetilde{\Sigma}$ of Σ corresponding to a finite normal cover $\Gamma_{\widetilde{\Sigma}} \to \Gamma_\Sigma$. Then for any C' with $\omega_{B(C')}$ balanced, by Lemma 6.4, up to an adjustment, the winding

numbers of the preimages of C' on each lift $\widetilde{\omega}_{B(C')}$ of $\omega_{B(C')}$ concentrate on one piece $\widetilde{C}'$ with $w(\widetilde{C}')$ divisible by $n(C')$. Hence for any piece C' other than C, every preimage $\widetilde{C}'$ on $\widetilde{\Sigma}$ has $w(\widetilde{C}')$ divisible by $n(C')$. Now a lift $\widetilde{P(C')}$ of $P(C)$ connects $\widetilde{C}'$ and some preimage $\widetilde{C}$ of C, by our choice of $n(C')$, the winding number at $\widetilde{C}'$ can be eliminated by an adjustment at the cost of changing $w(\widetilde{C})$ by a multiple of $|D_v|$.

In this way, all pieces of $\widetilde{\Sigma}$ are either genuine disks or a preimage of C that is disk-like. Since each lift of the cycle ω must be imbalanced, further applying Lemma 6.3, we may assume preimages of C to have large winding numbers that can be merged into a single preimage $\widetilde{C}$ with $w(\widetilde{C})$ divisible by $dm^N \ell^N$ after an adjustment. $\qquad\square$

Lemma 6.12 *For c reduced, if a component Σ of S contains an imbalanced cycle ω, then a connected finite cover of Σ is disk-only after a suitable adjustment.*

Proof Let $\widetilde{\Sigma}$ be the connected cover as in Lemma 6.11, which almost consists of genuine disks except for one piece $\widetilde{C}$ with $w(\widetilde{C})$ divisible by $dm^N \ell^N$, where N can be chosen arbitrarily large without changing the surface $\widetilde{\Sigma}$ by Lemma 6.3, since $\widetilde{C}$ sits on a lift of ω which is imbalanced. By Lemma 6.9, $\widetilde{\Sigma}$ contains an embedded imbalanced cycle ω_{im}. Since c is reduced, we have $\chi(\Gamma_S) = \widehat{\chi}(S) < 0$ by Proposition 6.8, so there must be an embedded cycle distinct from ω_{im}, and thus the conclusion follows from Lemma 6.5. $\qquad\square$

Corollary 6.13 *If no $g_i \in \underline{g}$ is t-balanced, then the reduced chain c has an extremal surface.*

Proof Let S be a simple relative admissible surface obtained from the optimal solution to the linear programming problem as in Theorem 5.10. It consists of disk-like pieces by Lemma 6.1. Every component Σ of S contains imbalanced loop since no $g_i \in \underline{g}$ is t-balanced, hence a connected finite cover of Σ can be made disk-only by Lemma 6.12. Taking the union of suitable copies of these finite covers of components produces an extremal surface for c. $\qquad\square$

Now we consider a component Σ without any imbalanced cycle. By Lemma 6.9, there is an orientation preserving homomorphism $\phi : \Gamma_\Sigma \to \widetilde{\Gamma}$. Any two such homomorphisms differ by a translation on $\widetilde{\Gamma}$.

Given ϕ, for each $k \in \mathbb{Z}$, let $s_k(\Sigma)$ be the sum of winding numbers of arcs on the boundary of all pieces C in Σ with $\phi(C) = k$. The quantity $s_k(\Sigma)$ is 0 for all but finitely many $k \in \mathbb{Z}$ since the graph Γ_Σ is finite. Let $s(\Sigma)$ be the sum of $s_k(\Sigma)(m/\ell)^k$ over all $k \in \mathbb{Z}$.

Lemma 6.14 *The number $s(\Sigma)$ defined above satisfies the following properties:*

(1) *It is invariant under adjustment on Σ;*

(2) *$s(\Sigma) = 0$ if Σ can be made disk-only by an adjustment; and*

(3) *Different choices of ϕ affect $s(\Sigma)$ by non-zero multiplicative scalars.*

Proof Suppose an edge e on Γ_Σ represents the gluing of two pieces $C = o(e)$ and $C' = t(e)$ on Σ along paired turns. Then we have $\phi(C') = \phi(C) + 1$. If we increase the winding number of the turn by 1 viewed from C, then $w(C)$ would increase by $o_e(1) = M$ and $w(C')$ would decrease (due to opposite orientation) by $t_e(1) = L$. Thus $s(\Sigma)$ increases by $M(m/\ell)^{\phi(C)} - L(m/\ell)^{\phi(C)+1} = 0$. This proves (1), from which (2) immediately follows. (3) is obvious since a different choice of ϕ differs by a constant k', which changes $s(\Sigma)$ by a scalar $(m/\ell)^{k'}$. $\qquad\qquad\square$

Lemma 6.15 *Suppose there is an orientation preserving homomorphism $\phi : \Gamma_\Sigma \to \widetilde{\Gamma}$. With the notation above, a connected finite cover of Σ can be made disk-only after a suitable adjustment if $s(\Sigma) = 0$.*

Proof Since Γ_Σ is finite and connected, the range of ϕ is $[N_{\min}, N_{\max}] \cap \mathbb{Z}$ for some integers $N_{\min}$ and $N_{\max}$. Let $N := N_{\max} - N_{\min}$. Then it is easy to see that, for any piece C with $w(C)$ divisible by $n := dm^N \ell^N$ and any path P from C to some C', we have $w(C) \in \mathrm{Dom}\,\alpha(P)$ and $\alpha(P)w(C) = w(C)(-\ell/m)^{\phi(C')-\phi(C)}$.

For any C, choose a boundary component $B(C)$ passing through C. By Lemma 6.10, there is a finite cover $\widetilde{\Sigma}$ of Σ corresponding to a connected finite normal cover of Γ_Σ where each lift $\widetilde{\omega}_{B(C)}$ of $\omega_{B(C)}$ has degree divisible by n, for any piece C in Σ. Hence by Lemma 6.4, after an adjustment every piece $\widetilde{C}$ of $\widetilde{\Sigma}$ has winding number $w(\widetilde{C})$ divisible by n. Composing the covering projection with ϕ, we have an orientation preserving homomorphism $\widetilde{\phi} : \Gamma_{\widetilde{\Sigma}} \to \widetilde{\Gamma}$ with the same range as ϕ. Thus the observation made in the first paragraph also applies to $\Gamma_{\widetilde{\Sigma}}$, and all winding numbers can be merged into a single piece $\widetilde{C}_*$ of $\widetilde{\Sigma}$ up to an adjustment. Note that $s(\widetilde{\Sigma})$ is simply $s(\Sigma)$ multiplied by the covering degree, so it must also vanish. It follows that $\widetilde{C}_*$ is a genuine disk since all other pieces are. $\qquad\qquad\square$

Note that $g \in \mathrm{BS}(M, L) = \langle a, t \in \mathbb{Z} \mid a^M = ta^L t^{-1} \rangle$ is t-balanced if and only if it lies in $\ker h$, which has presentation $\langle a_k, k \in \mathbb{Z} \mid a_k^M = a_{k+1}^L \rangle$ with inclusion into $\mathrm{BS}(M, L)$ given by $a_k \mapsto t^k a t^{-k}$. Hence each t-balanced g can be written as $\prod_j a_{k_j}^{u_j}$, where $k_j, u_j \in \mathbb{Z}$. Let $s(g) := \sum_j u_j (m/\ell)^{k_j}$. Note that whether $s(g) = 0$ only depends on the conjugacy class of g since $s(wgw^{-1}) = s(g)(m/\ell)^{h(w)}$ for any $w \in \mathrm{BS}(M, L)$.

Corollary 6.16 Let $M \neq \pm L$. For the reduced chain c above, if every t-balanced $g_i \in \underline{g}$ satisfies $s(g_i) = 0$, then an extremal surface for c exists.

Proof Let S be a relative admissible surface built out of the optimal solution to the linear programming in Theorem 5.10. Components containing imbalanced cycles can be made extremal after taking a finite cover by Lemma 6.12. If Σ is a component without imbalanced loop, fix some ϕ provided by Lemma 6.9, note that $s(\Sigma)$ can be computed by a sum over boundary components B, where arcs on B contribute $ks(g_i)(m/\ell)^n$ for some $n \in \mathbb{Z}$ depending on ϕ if B represents g_i^k, where g_i is necessarily t-balanced and $s(g_i) = 0$ by our assumption. Thus $s(\Sigma) = 0$. By Lemma 6.15, a finite cover of Σ can be made extremal. $\square$

Remark 6.17 Following the comment at the beginning of this subsection, it is easy to prove similar results for the exceptional cases.

(1) If $M = L$, the number $s(g) := \sum u_j$ is well defined for any $g = \prod_j a^{u_j} t^{\epsilon_j}$ ($\epsilon_j = \pm 1$). The reduced chain c bounds an extremal surface if $s(g_i) = 0$ for all $g_i \in \underline{g}$.

(2) If $M = -L$, the number $s(g) := \sum (-1)^j u_j$ is well defined for $g = \prod_j a^{u_j} t^{\epsilon_j}$ ($\epsilon_j = \pm 1$) if $h(g)$ is even. The reduced chain c bounds an extremal surface if either $h(g_i)$ is odd or $s(g_i) = 0$ for all $i \in \underline{g}$.

Remark 6.18 Based on this result, the method in [9] produces (closed) surface subgroups in certain graphs of groups.

Corollary 6.19 Let G be a graph of groups with each vertex group isomorphic to $BS(M, L)$ for $M \neq \pm L$ and edge groups isomorphic to $\mathbb{Z}$. Assume for each vertex group the inclusions of adjacent edge groups do not form pseudo-inverses except the identity, where each t-balanced word g has $s(g) = 0$. Then any non-trivial class in $H_2(G; \mathbb{Q})$ has an integer multiple represented by a π_1-injective surface of genus at least 1.

Now we upgrade Lemma 6.12 and Lemma 6.15 to characterize the existence of extremal surfaces for c in term of data canonically obtained from the result of the linear programming problem computing $\mathrm{scl}_{BS(M,L)}(c)$. Such data is expressed as branched surfaces which we describe as follows.

Recall from Lemma 5.8 that $\mathrm{scl}_{BS(M,L)}(c)$ is computed as the minimum of a convex piecewise rational linear function on a compact rational polyhedron $\mathcal{C}(c)$. Thus the
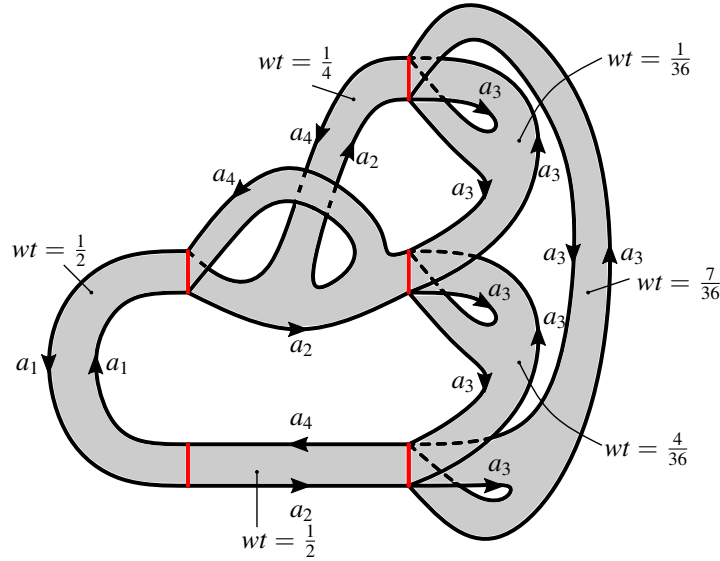
Figure 10: This is a balanced branched surface consisting of 6 pieces with gluing loci colored red. It carries the surface admissible for $[a, t^2]$ constructed in the proof of Proposition 6.28.

set of points $x \in \mathcal{C}(c)$ achieving the optimal value is a nonempty (finite-sided) convex rational polyhedron $\mathcal{P}_{opt}$. For each rational point $x \in \mathcal{P}_{opt}$, by Lemma 6.1 and Lemma 5.7, there are *finitely* many disk-like pieces $\{C_j\}$ and weights $wt_j \in \mathbb{Q}_{>0}$ such that

$$(6\text{–}5) \qquad \sum_j wt_j x(C_j) = x \quad \text{and} \quad \sum_j wt_j = \kappa_v(x).$$

Let $\{C_j\}_{j \in J}$ be the finite set of disk-like pieces that possibly appear in (6–5) for some $x \in \mathcal{P}_{opt}$. Then the set of weights satisfying equation (6–5) for some $x \in \mathcal{P}_{opt}$ form a convex rational polyhedron $\mathcal{Q}_w$ in $\mathbb{R}_{\geq 0}^J$.

For each weight $wt \in \mathcal{Q}_w$, we obtain a simple branched surface S from the disk-like pieces C_j with positive weights: this is done by gluing each turn to all its paired turns as different branches. Then each piece C_j embeds into S, and branching only occurs at gluing loci. See Figure 10 for an example.

Such a branched surface only depends on the open face $\mathcal{F}$ of $\mathcal{Q}_w$ in which wt lies, so we denote it by $S(\mathcal{F})$.

Each gluing locus has a co-orientation induced by the orientation of Γ. There is a projection $\pi : S(\mathcal{F}) \to \Gamma$ taking each piece to the unique vertex of Γ and each gluing locus to the edge e of Γ so that the co-orientation agrees with the orientation of e.

$S(\mathcal{F})$ may not be connected. A component $\Sigma(\mathcal{F})$ of $S(\mathcal{F})$ is *balanced* if π lifts to $\widetilde{\Gamma}$. Explicitly, $\Sigma(\mathcal{F})$ is balanced if every loop ω in it intersects an even number of gluing loci, exactly half of which have co-orientations consistent with the orientation of ω. For instance, the branched surface in Figure 10 is balanced.

For a balanced component $\Sigma(\mathcal{F})$, let ϕ be a lift of π. Given a weight $wt \in \mathcal{F}$, let $s(\Sigma(\mathcal{F}), wt)$ be the sum of $wt_j s_j (m/\ell)^{\phi(C_j)}$ over all $j \in J$, where s_j is the sum of winding numbers of arcs on the boundary of C_j. Whether $s(\Sigma(\mathcal{F}), wt)$ vanishes does not depend on the choice of ϕ.

Finally we are in a position to state our criterion for the existence of extremal surfaces in terms of such branched surfaces.

Theorem 6.20 (extremal surfaces) *Let $M \neq \pm L$. A reduced rational chain c has an extremal surface if and only if there is some wt in an open face $\mathcal{F}$ of $\mathcal{Q}_w$, such that each balanced component $\Sigma(\mathcal{F})$ of $S(\mathcal{F})$ has $s(\Sigma(\mathcal{F}), wt) = 0$.*

Remark 6.21 This criterion can be checked by an algorithm. For each open face $\mathcal{F}$, each equation $s(\Sigma(\mathcal{F}), wt) = 0$ is rational linear in wt. Thus the existence of wt in a given $\mathcal{F}$ satisfying the criterion is a linear programming (feasibility) problem, and there are only finitely many open faces $\mathcal{F}$ of $\mathcal{Q}_w$ to enumerate.

To prove Theorem 6.20, in view of Lemma 6.12 and Lemma 6.15, it suffices to produce surfaces without any branching satisfying analogous conditions.

Given a rational weight $wt \in \mathcal{Q}_w$, a surface Σ in simple normal form is *carried by* $\Sigma(\mathcal{F})$ with weight wt if there is some $N \in \mathbb{Z}_+$ such that Σ is made of Nwt_j copies of C_j for each $j \in J$. The extremal surfaces S that we look for consist of components Σ, each carried by some $\Sigma(\mathcal{F})$. Note that if $\Sigma(\mathcal{F})$ is balanced, then there is an orientation preserving homomorphism $\phi : \Gamma_\Sigma \to \widetilde{\Gamma}$ sending each copy of C_j to $\phi(C_j)$ where ϕ is a lift of π. In this case, we have $s(\Sigma) = Ns(\Sigma(\mathcal{F}), wt)$.

We can actually find Σ that is connected and has the same balanced properties as $\Sigma(\mathcal{F})$. This is Lemma 6.23 below. To prove it, recall that an undirected graph Λ is strongly connected (or 2-edge-connected) if every edge is non-separating. A strongly connected component (SCC for short) of Λ is a maximal strongly connected subgraph. Any connected graph Λ uniquely decomposes as a tree of its SCCs. A SCC is trivial if it consists of a single vertex.

Lemma 6.22 *Suppose Λ is a connected graph where each vertex has valence at least 2. Then Λ has a strongly connected double cover $\widetilde{\Lambda}$.*
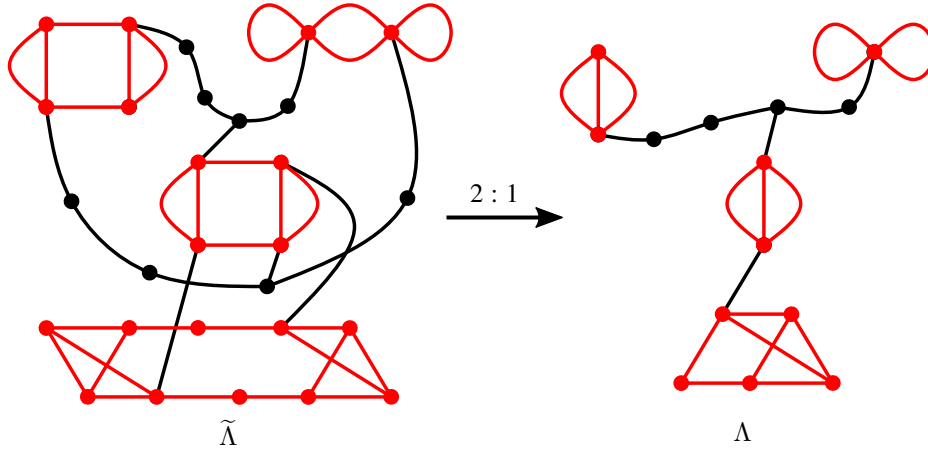
Figure 11: The graph Λ has its non-trivial SCCs colored red. The graph $\widetilde{\Lambda}$ is a strongly connected double cover of Λ.

Proof Express Λ as a tree of its SCCs. Our assumption implies that each leaf of the tree must be a non-trivial SCC. Let $C_1, \ldots, C_k$ be the non-trivial SCCs of Λ. Then their complement is a union of embedded trees $T_1, \ldots, T_n$ in Λ, where each leaf of a tree T_i is glued to a vertex in some C_j. Take a connected double cover $\widetilde{C}_i$ of each non-trivial SCC C_i and take two copies $\widetilde{T}_i^{(1)}, \widetilde{T}_i^{(2)}$ of each embedded tree T_i. They can be assembled into a double cover $\widetilde{\Lambda}$ of Λ: For each leaf v on T_i that is glued to a vertex u on some C_j, match the two leaves $\tilde{v}^{(1)}, \tilde{v}^{(2)}$ on $\widetilde{T}_i^{(1)}, \widetilde{T}_i^{(2)}$ corresponding to v with the two preimages $\tilde{u}^{(1)}, \tilde{u}^{(2)}$ of u on $\widetilde{C}_j$. See Figure 11 for an illustration. Then the graph $\widetilde{\Lambda}$ obtained is strongly connected. □

Lemma 6.23 *For each rational weight wt in an open face $\mathcal{F}$, each component $\Sigma(\mathcal{F})$ carries a connected surface Σ with weight wt. Moreover, we may choose Σ to contain an imbalanced loop unless $\Sigma(\mathcal{F})$ is balanced.*

Proof Let Σ be any surface carried by $\Sigma(\mathcal{F})$ with the given weight. First we show that for any given path P of pieces in $\Sigma(\mathcal{F})$, we can modify Σ to contain a lift of P without increasing the number of components of Σ. We accomplish this by lifting longer subpaths on P. At each step, suppose our lift ends with a piece $\widetilde{C}_1$ corresponding to C_1 on P and let $\widetilde{C}_2$ be an arbitrarily chosen lift of the piece C_2 next to C_1 on P extending the subpath. For $i = 1, 2$, let $\widetilde{C}_i'$ be the piece on Σ glued to $\widetilde{C}_i$ along the turn corresponding to the turn on $\Sigma(\mathcal{F})$ along which C_1 and C_2 are glued. Now we modify Σ by gluing $\widetilde{C}_1$ to $\widetilde{C}_2$ and $\widetilde{C}_2'$ to $\widetilde{C}_1'$ instead. Figure 12 illustrates this at the
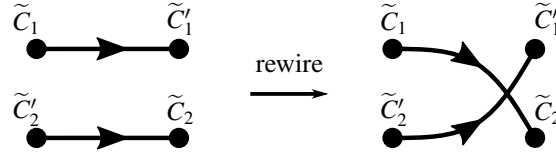
Figure 12: The two edges on the left represent pairings of turns of the same type. The rewiring switches the end points of the edges, ie the gluing of pieces.

level of graphs. Such a rewiring operation extends our lift. Moreover, if at least one of the two edges representing the original gluing in Γ_Σ is non-separating, then the number of components in Σ does not increase, and it decreases if in addition the two edges lie in different components of Σ. Finally note that we may assume both edges to be non-separating up to taking a double cover of Σ by Lemma 4.11 and Lemma 6.22.

Now let Σ be a surface carried by $\Sigma(\mathcal{F})$ with the given weight that has the minimal number of components. Suppose Σ is disconnected and let $\widetilde{C}_0$ and $\widetilde{C}_2$ be two pieces in different components of Σ. Connect the corresponding pieces C_0 and C_2 in $\Sigma(\mathcal{F})$ by a path P, and let C_1 be the piece next to C_2 on P. Modify Σ to lift P by the process above until we obtain a lift of the subpath from C_0 to C_1, and proceed the last step by choosing $\widetilde{C}_2$ to be the lift of C_2. This decreases the number of components of Σ by at least one since the components containing $\widetilde{C}_0$ and $\widetilde{C}_2$ are merged at a certain stage of this process. This contradicts the choice of Σ and thus Σ must be connected.

For Σ connected, since $\Sigma(\mathcal{F})$ contains an imbalanced cycle unless it is balanced, we can obtain a copy of such a cycle in Σ applying the rewiring operations above.   □

Proof of Theorem 6.20 Suppose S is an extremal surface. Then it is carried by some $S(\mathcal{F})$ with weight $wt \in \mathcal{Q}_w$, where $\mathcal{F}$ is the open face containing wt. Since each piece of S is a genuine disk, each component Σ carried by a balanced component $\Sigma(\mathcal{F})$ has $s(\Sigma) = 0$. We conclude that $s(\Sigma(\mathcal{F}), wt) = 0$ for any balanced component $\Sigma(\mathcal{F})$ since the sum of $s(\Sigma)$ over all components Σ of S carried by $\Sigma(\mathcal{F})$ is a positive multiple of $s(\Sigma(\mathcal{F}), wt)$.

Conversely, suppose there is a weight wt in an open face $\mathcal{F}$ satisfying the criterion. Since each equation $s(\Sigma(\mathcal{F}), wt) = 0$ is rational linear in wt, by rationality of $\mathcal{F}$, we may assume wt to be rational. By Lemma 6.23, each component $\Sigma(\mathcal{F})$ carries a connected surface Σ with the given weight, such that Γ_Σ contains no imbalanced loop if and only if $\Sigma(\mathcal{F})$ is balanced. Then each Σ corresponding to a balanced $\Sigma(\mathcal{F})$ has $s(\Sigma) = 0$ since $s(\Sigma(\mathcal{F}), wt) = 0$. By Lemma 6.12 and Lemma 6.15, an extremal surface exists.   □

6.3 Lower bounds from duality

To obtain explicit formulas for scl, especially when we consider chains in $\mathrm{BS}(M, L)$ with parameters or with varying M and L, it is often too complicated to work out the linear programming problems. Proving a sharp lower bound is usually the main difficulty. The classical approach using Bavard duality relies on finding extremal quasimorphisms, which is quite hard in practice.

A new method using the idea of linear programming duality is used in [18] to obtain lower bounds in free products. It actually applies to graphs of groups and proves the so-called spectral gap properties with sharp estimates, which is discussed in detail in a paper with Nicolaus Heuer [19]. Here we simply describe this method in our setting. Fix any one of the two setups which determines the notion of disk-like vectors as in Subsection 6.1.

To each turn $(a_i, \bar{w}, a_j)$ we assign a *non-negative cost* $q_{i,\bar{w},j}$. This defines a linear cost function q on $\mathcal{C}_v$. In particular, the cost of a piece is the sum of the costs of the turns on its polygonal boundary.

Recall from Lemma 5.8 and Remark 5.9 that computing scl is equivalent to maximizing the function κ_v counting the (normalized) number of disk-like pieces on $\mathcal{C}(c)$, where $c = \sum r_i g_i$. Hence giving upper bounds of κ_v produces lower bounds of scl.

Lemma 6.24 *If $q(d) \geq 1$ for any disk-like vector d, then $\kappa_v(x) \leq q(x)$ for any $x \in \mathcal{C}_v$.*

Proof For any admissible expression $x = x' + \sum t_i d_i$ with d_i disk-like, $t_i \geq 0$, and $x' \in \mathcal{C}_v$, we have $q(x) = q(x') + \sum t_i q(d_i) \geq \sum_i t_i$. Thus $q(x) \geq \kappa_v(x)$. □

For each vector $x \in \mathcal{C}(c)$, we write it as $\sum t_{i,\bar{w},j}(a_i, \bar{w}, a_j)$ where $t_{i,\bar{w},j}$ is the coordinate corresponding to the basis element $(a_i, \bar{w}, a_j) \in T_v$. We think of $t_{i,\bar{w},j}$ as the normalized number of turns of type $(a_i, \bar{w}, a_j)$. Recall from Section 5 that the gluing condition requires $t_{i,\bar{w},j} = t_{i',\bar{w}',j'}$ if $(a_i, \bar{w}, a_j)$ and $(a_{i'}, \bar{w}', a_{j'})$ are paired triples, and the normalizing condition implies $\sum_{\bar{w},j} t_{i,\bar{w},j} = r_k$ if $a_i \subset \gamma_k$ and $\sum_{i,\bar{w}} t_{i,\bar{w},j} = r_k$ if $a_j \subset \gamma_k$.

If we can choose the costs so that

(1) $q(d) \geq 1$ for any disk-like vector d, and

(2) for any $x \in \mathcal{C}(c)$ expressed in the form above, $q(x) = \sum q_{i,\bar{w},j} t_{i,\bar{w},j}$ is equal to or bounded above by a constant K on $\mathcal{C}(c)$ by the gluing and normalizing conditions,

then Lemma 6.24 implies $\kappa_v \leq K$ on $\mathcal{C}(c)$. Combining with Lemma 5.8 and Remark 5.9, this gives a lower bound of scl.
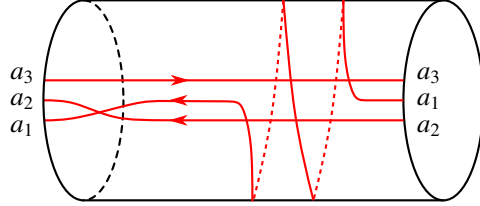
Figure 13: The three arcs of $a^k t^2 + 2t^{-1}$ in the thickened vertex space $N(X_v)$ with $k = 2$.

6.4 Examples with explicit formulas

In this subsection, we compute three examples of complexities $\rho = 0, 1, 2$ that are not t-alternating words. As we will see, the computations get more complicated as the complexity increases.

Proposition 6.25 *For $d = \gcd(|M|, |L|)$, we have*

$$\mathrm{scl}_{\mathrm{BS}(M,L)}(a^k t^2 + 2t^{-1}) = \frac{1}{2} - \frac{\gcd(|k|, d)}{2d}.$$

Proof Let $n_k = \frac{d}{\gcd(|k|,d)}$, which is the order of $[k]$ in $\mathbb{Z}/d\mathbb{Z}$. We use the notation introduced in Subsection 6.1.

Let γ_1 be the tight loop representing $g_1 = a^k t^2$ consisting of two arcs a_1 and a_2 with winding numbers k and 0 respectively. Let γ_2 be the tight loop representing $g_2 = t^{-1}$ consisting of a single arc a_3 with winding number 0. The three arcs are depicted in Figure 13. By formula (6–1), we have $W_0(a_1) = W_0(a_2) = W_0(a_3) = d\mathbb{Z}$. Then Setup 1 and Setup 2 coincide, which we use. We have $\rho(g_1 + 2g_2) = \rho(g_1) = \rho(g_2) = 0$ and $D_v = d\mathbb{Z}$. As a consequence, we have $W_e = \{1\}$. That is, we ignore the winding numbers of turns and use a pair (a_i, a_j) instead of a triple $(a_i, \bar{w}, a_j)$ to represent a turn.

We have a turn (a_1, a_3) paired with (a_3, a_2), and a turn (a_2, a_3) paired with (a_3, a_1). The defining equation $\partial = 0$ implies that $\mathcal{C}_v$ consists of vectors of the form $\xi(x, y) = x(a_1, a_3) + x(a_3, a_1) + y(a_2, a_3) + y(a_3, a_2)$ with $(x, y) \in \mathbb{R}^2_{\geq 0}$, which has winding number $kx \bmod d$. Thus such a vector is disk-like if and only if $(x, y) \neq (0, 0) \in \mathbb{Z}^2_{\geq 0}$ and $kx \in d\mathbb{Z}$. This describes the set $\mathcal{D}(v)$ of disk-like vectors, from which we get $\mathcal{D}(v) + \mathcal{C}_v = \{\xi(n_k, 0), \xi(0, 1)\} + \mathcal{C}_v$ (See Figure 14). It follows that $\kappa_v(\xi(x, y)) = x/n_k + y$. For the chain $c = g_1 + 2g_2$, the normalizing condition requires $x = y = 1$, so $\xi(1, 1)$ is the only vector in $\mathcal{C}(c)$. Thus by Lemma 5.8 and Remark 5.9, we have

$$\mathrm{scl}(c) = 1 - \frac{1}{2}\kappa_v(\xi(1, 1)) = \frac{1}{2} - \frac{1}{2n_k}.$$
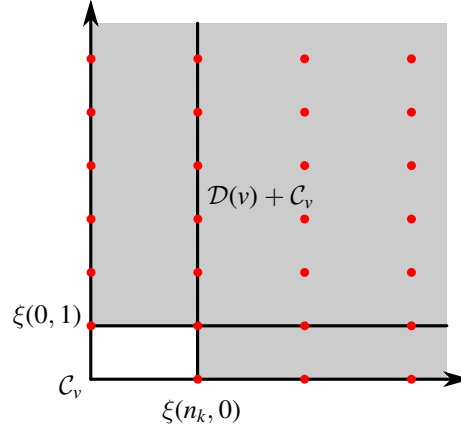
Figure 14: The red dots are disk-like vectors in $\mathcal{D}(v)$. The gray region is the set $\mathcal{D}(v) + \mathcal{C}_v$.

<div style="text-align: right;">□</div>

Proposition 6.26 *For all $|M|, |L| \geq 2$, we have*

$$\mathrm{scl}_{BS(M,L)}(atta^{-1}t^{-1} + t^{-1}) \leq \frac{1}{2} - \frac{1}{4|M|} - \frac{1}{4|L|}.$$

The equality holds if $d = \gcd(|M|, |L|)$ satisfies $d \geq \frac{|M|+|L|}{2\min\{|M|,|L|\}}$.

Proof We have 3 arcs a_1, a_2, a_3 on the loop γ_1 representing $atta^{-1}t^{-1}$ with winding numbers $1, 0, -1$ respectively, and have another arc a_4 on the other loop γ_2 with winding number 0. It easily follows from the definitions in Subsection 6.1 that $\rho(\gamma_1) = 1$, $\rho(\gamma_2) = 0$, so our chain c has complexity $\rho(c) = 1$.

We use Setup 1 in this proof. By formula (6–1), we have $W_0(a_1) = dm\mathbb{Z}$, $W_0(a_2) = W_0(a_4) = d\mathbb{Z}$, $W_0(a_3) = d\ell\mathbb{Z}$. Hence $D_v = dm\ell\mathbb{Z}$ and $W_e = \mathbb{Z}/m\ell\mathbb{Z}$. Turns are paired up as follows:

$$(a_1, \bar{w}, a_1) \quad \leftrightarrow \quad (a_3, -\bar{w}, a_2) \qquad (a_1, \bar{w}, a_4) \quad \leftrightarrow \quad (a_4, -\bar{w}, a_2)$$

$$(a_2, \bar{w}, a_4) \quad \leftrightarrow \quad (a_4, -\bar{w}, a_3) \qquad (a_2, \bar{w}, a_1) \quad \leftrightarrow \quad (a_3, -\bar{w}, a_3)$$

These are the only turns, so each piece C falls into exactly one of the following three types.

(i) The polygonal boundary contains both a_2 and a_4. It is disk-like if and only if $w(C)$ is divisible by d since $W(a_2) = d\mathbb{Z}$. Changing the winding numbers of turns will change $w(C)$ by a multiple of $|M|$ or $|L|$, both divisible by d.

(ii) The polygonal boundary contains a_1 only. Then the winding number $w(C) \equiv k$ mod $|M|$ if there are k copies of a_1 on the boundary, which does not depend on the winding numbers of turns. Thus it is disk-like if and only if $w(C) \in |M|\mathbb{Z} = W(a_1)$, ie k is divisible by $|M|$.

(iii) The polygonal boundary contains a_3 only. Similar to the previous case, it is disk-like if and only if the number of copies of a_3 on the boundary is divisible by $|L|$.

In summary, whether a piece is disk-like does not depend on the winding numbers of turns on its polygonal boundary. Thus we simply assume all turns to have winding numbers 0 in what follows.

We prove the upper bound by constructing a simple relative admissible surface S of degree $|2ML|$ consisting of the following disk-like pieces described by the turns on the polygonal boundaries (also see Figure 15).

(1) $(a_2, 0, a_4) + (a_4, 0, a_2)$, take $|ML|$ copies of this piece;

(2) $(a_2, 0, a_1) + (a_1, 0, a_4) + (a_4, 0, a_3) + (a_3, 0, a_2)$, take $|ML|$ copies of this piece;

(3) $|M|(a_1, 0, a_1)$, take $|L|$ copies of this piece;

(4) $|L|(a_3, 0, a_3)$, take $|M|$ copies of this piece.

It is easy to check that these are disk-like pieces and that the gluing conditions hold. With notation as in Lemma 4.15, we get $\widehat{\chi}(S) = -2|ML| + |M| + |L|$ and obtain the upper bound

$$\mathrm{scl}_{\mathrm{BS}(M,L)}(atta^{-1}t^{-1} + t^{-1}) \leq \frac{1}{2} - \frac{1}{4|M|} - \frac{1}{4|L|}.$$

To obtain the lower bound, we use the duality method introduced in Subsection 6.3. Assign a cost q_{ij} independent of the winding number $\bar{w}$ to each turn $(a_i, \bar{w}, a_j)$ as in the following matrix $Q = (q_{ij})$, where $*$ appears if there are no such turns.

$$Q = \begin{pmatrix} \frac{1}{|M|} & * & * & 1 - \left(\frac{1}{|M|} + \frac{1}{|L|}\right) \\ \frac{1}{|M|} & * & * & 1 - \frac{1}{2}\left(\frac{1}{|M|} + \frac{1}{|L|}\right) \\ * & \frac{1}{|L|} & \frac{1}{|L|} & * \\ * & \frac{1}{2}\left(\frac{1}{|M|} + \frac{1}{|L|}\right) & 0 & * \end{pmatrix}$$

We first check that every disk-like piece costs at least 1 when $d \geq \frac{|M|+|L|}{2\min\{|M|,|L|\}}$. According to the classification of pieces above, only those of type (i) requires some attention. Let C be such a piece, which must contain a turn ending at a_4. Assume $|M| \leq |L|$.
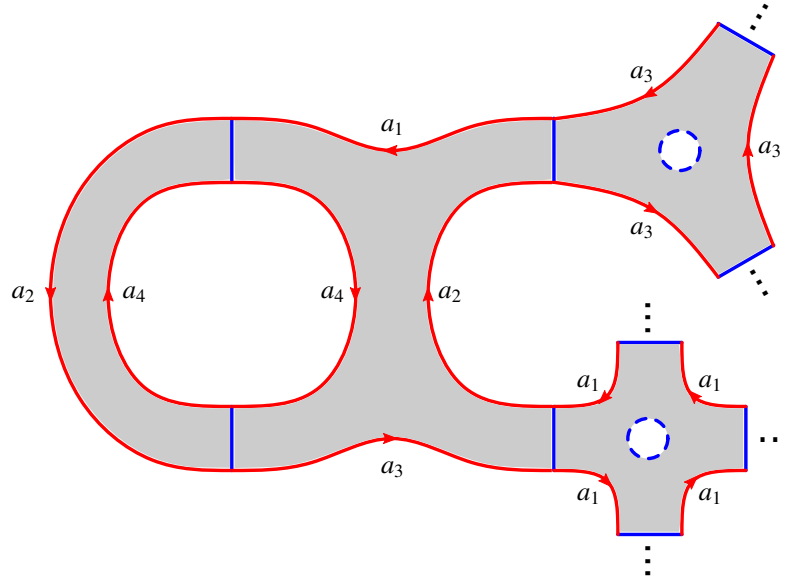
Figure 15: Part of a relative admissible surface involving the four disk-like pieces constructed to give the upper bound, illustrating the case $M = 4$ and $L = 3$

(1) Suppose C contains a turn from a_1 to a_4.

  (a) If this turn is followed by another from a_4 to a_3, then we must also have a turn from a_3 to a_2 (to leave a_3) and another from a_2 to a_1 so that the boundary closes up. In this case, the cost is at least $q_{14} + q_{43} + q_{32} + q_{21} = 1$.

  (b) If this turn is followed by another from a_4 to a_2 instead, then the cost is at least $q_{14} + q_{42} + \min(q_{21}, q_{24}) \geq 1$ since $2 \leq |M| \leq |L|$.

(2) Suppose C does not contain a turn from a_1 to a_4. Then C does not visit a_1 and must contain a turn from a_2 to a_4. If C also contains a turn from a_4 to a_2, then the cost will be at least $q_{24} + q_{42} = 1$. Otherwise, C is encoded as $n_1(a_2, 0, a_4) + n_1(a_4, 0, a_3) + n_2(a_3, 0, a_3) + n_1(a_3, 0, a_2)$ with integers $n_1 \geq 1$, $n_2 \geq 0$, and costs $n_1(q_{24} + q_{43} + q_{32}) + n_2 q_{33}$. Then C has winding number $w(C) \equiv -(n_1 + n_2) \mod d$. For C to be disk-like, we have $n_1 + n_2 \geq d$. Note that $q_{24} \geq 1/2 \geq q_{33}$ since $|M|, |L| \geq 2$. Therefore, the cost

$$n_1(q_{24} + q_{43} + q_{32}) + n_2 q_{33} \geq q_{24} + q_{43} + q_{32} + (d-1)q_{33} = 1 + \frac{d - \frac{1}{2}}{|L|} - \frac{1}{2|M|} \geq 1$$

since $d \geq \frac{|M| + |L|}{2|M|}$.

The other case $|M| \geq |L|$ is similar.

Now let $t_{ij} = \sum_{\bar{w}} t_{i,\bar{w},j}$, where $t_{i,\bar{w},j}$ is the normalized number of the turn $(a_i, \bar{w}, a_j)$. Then we obtain $t_{14} = t_{42}$ from the gluing conditions, which implies the total cost

$$\sum_{i,j} q_{ij}t_{ij} = \left[1 - \frac{1}{2}\left(\frac{1}{|M|} + \frac{1}{|L|}\right)\right](t_{14} + t_{24}) + \frac{1}{|M|}(t_{11} + t_{21}) + \frac{1}{|L|}(t_{32} + t_{33}).$$

The normalizing conditions imply $t_{14} + t_{24} = t_{11} + t_{21} = t_{32} + t_{33} = 1$ and thus

$$\sum_{i,j} q_{ij}t_{ij} = 1 + \frac{1}{2}\left(\frac{1}{|M|} + \frac{1}{|L|}\right),$$

which is an upper bound of $\kappa_v(x)$ for all $x \in \mathcal{C}(c)$ by Lemma 6.24. Hence

$$\mathrm{scl}_{\mathrm{BS}(M,L)}(atta^{-1}t^{-1} + t^{-1}) \geq \frac{1}{2} - \frac{1}{4|M|} - \frac{1}{4|L|}$$

by Lemma 5.8 and Remark 5.9. □

Remark 6.27 A slightly weaker lower bound

$$\mathrm{scl}_{\mathrm{BS}(M,L)}(atta^{-1}t^{-1} + t^{-1}) \geq \frac{1}{2} - \frac{1}{2\min\{|M|,|L|\}}.$$

holds for all $|M|, |L| \geq 2$, which can be proved in a similar way with much simpler computations using cost matrix

$$Q = \begin{pmatrix} \frac{1}{\min\{|M|,|L|\}} & * & * & 0 \\ 0 & * & * & 0 \\ * & 0 & \frac{1}{\min\{|M|,|L|\}} & * \\ * & 1 & 1 & * \end{pmatrix}.$$

This bound is sharp when $d = 1$ by constructing admissible surfaces in simple normal form.

In contrast to the two examples above, the winding numbers of turns cannot be ignored in the following example, which has higher complexity.

Proposition 6.28

$$\mathrm{scl}_{\mathrm{BS}(2,3)}([a, t^2]) = \frac{5}{24}.$$

Proof We have four arcs $a_1, \ldots, a_4$ with winding numbers $1, 0, -1, 0$ respectively. We use Setup 1 and compute $W(a_1) = 4\mathbb{Z}$, $W(a_2) = W(a_4) = 6\mathbb{Z}$, and $W(a_3) = 9\mathbb{Z}$.

Then $D_v = D_e = 36\mathbb{Z}$, $W_e = \mathbb{Z}/36\mathbb{Z}$ and the complexity $\rho(c) = 2$. The turns are paired up as follows.

$$
\begin{aligned}
(a_1, \bar{w}, a_1) &\leftrightarrow (a_4, -\bar{w}, a_2) \\
(a_2, \bar{w}, a_4) &\leftrightarrow (a_3, -\bar{w}, a_3) \\
(a_2, \bar{w}, a_1) &\leftrightarrow (a_4, -\bar{w}, a_3) \\
(a_1, \bar{w}, a_4) &\leftrightarrow (a_3, -\bar{w}, a_2)
\end{aligned}
$$

To get $5/24$ as an upper bound, we present a simple relative admissible surface S of degree 36 consisting of the following disk-like pieces described by the turns on the polygonal boundaries. By the computation above and our orientation on e, the maps $\overline{o_e}, \overline{t_e} : \mathbb{Z}/36\mathbb{Z} \to \mathbb{Z}/36\mathbb{Z}$ are given by $\overline{o_e}(\bar{w}) = 2\bar{w}$ and $\overline{t_e}(\bar{w}) = 3\bar{w}$.

(1) $(a_1, 0, a_1) + (a_1, 1, a_1)$, which is disk-like since its winding number $2w(a_1) + \overline{o_e}(0 + 1) \in 4 + D_v$ lies in $W(a_1) = 4\mathbb{Z}$. This is the leftmost piece in Figure 10. Take 18 copies of this piece;

(2) $(a_2, 0, a_4) + (a_4, 0, a_2)$, which is disk-like since its winding number $w(a_2) + w(a_4) + \overline{o_e}(0) + \overline{t_e}(0) \equiv 0 \mod 36$ lies in $W(a_2) = 6\mathbb{Z}$. This is depicted in the bottom-middle of Figure 10. Take 18 copies of this piece.

(3) $(a_2, 1, a_4) + (a_2, 2, a_4) + 2(a_4, -1, a_2)$, which is disk-like since its winding number $2w(a_2) + 2w(a_4) + \overline{o_e}(1 + 2) + 2\overline{t_e}(-1) \equiv 0 \mod 36$ lies in $W(a_2) = 6\mathbb{Z}$. This is depicted in the upper-middle of Figure 10. Take 9 copies of this piece.

(4) $(a_3, 0, a_3) + 2(a_3, -1, a_3)$, which is disk-like since its winding number $3w(a_3) + \overline{t_e}(2(-1)) \equiv -9 \mod 36$ lies in $W(a_3) = 9\mathbb{Z}$. This is the piece on the right of Figure 10 with weight $4/36$. Take 4 copies of this piece.

(5) $(a_3, -2, a_3) + 2(a_3, 0, a_3)$, which is disk-like since its winding number $3w(a_3) + \overline{t_e}(-2) \equiv -9 \mod 36$ lies in $W(a_3) = 9\mathbb{Z}$. This is the piece on the right of Figure 10 with weight $7/36$. Take 7 copies of this piece.

(6) $(a_3, -1, a_3) + 2(a_3, -2, a_3)$, which is disk-like since its winding number $3w(a_3) + \overline{t_e}(-1 - 2 - 2) \equiv -18 \mod 36$ lies in $W(a_3) = 9\mathbb{Z}$. This is the piece on the right of Figure 10 with weight $1/36$. Take 1 copy of this piece.

It is easy to see that the gluing condition is satisfied and the surface S obtained is carried by the branched surface in Figure 10. Note that S is relative admissible of degree 36 with $\widehat{\chi}(S)/36 = 2 - 57/36 = 5/12$, and thus

$$
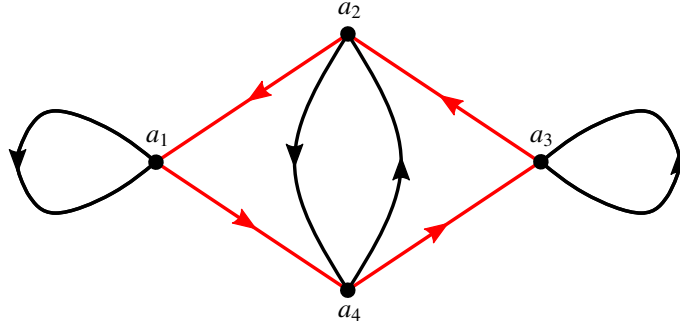\mathrm{scl}_{\mathrm{BS}(2,3)}([a, t^2]) \leq \frac{5}{24}.
$$

Figure 16: The graph Y indicating turns between arcs, except that every single edge represents $|W_e| = 36$ multiedges

To establish the lower bound, assign cost $q_{i,\bar{w},j}$ to the turn $(a_i, \bar{w}, a_j)$ with

$$q_{1,\bar{w},1} = \begin{cases} 1/4 & \text{if } \bar{w} \text{ is even}, \\ 3/4 & \text{otherwise}, \end{cases} \qquad q_{4,\bar{w},2} = \begin{cases} 1 & \text{if } \bar{w} \text{ is even}, \\ 1/2 & \text{otherwise}, \end{cases}$$

and for any $\bar{w}$

$$q_{2,\bar{w},4} = 0, \quad q_{3,\bar{w},3} = 1/3, \quad q_{2,\bar{w},1} = 1/3, \quad q_{4,\bar{w},3} = 0, \quad q_{1,\bar{w},4} = 1/4, \quad q_{3,\bar{w},2} = 1.$$

We check each disk-like piece C costs at least 1.

(1) Suppose C does not contain any turns of forms $(a_2, \bar{w}, a_1)$, $(a_4, \bar{w}, a_3)$, $(a_1, \bar{w}, a_4)$, or $(a_3, \bar{w}, a_2)$. That is, the red turns in Figure 16 are excluded. Then there are three cases:

    (a) The boundary only contains a_1. By formula (6–3), $w(a_1, \bar{w}, a_1) = w(a_1) + \overline{o_e}(\bar{w}) = 1 + 2\bar{w} \in \mathbb{Z}/36\mathbb{Z}$. Thus we have $w(C) \equiv n_0 + 3n_1 \mod 4$, where n_0 (resp. n_1) is the total number of turns $(a_1, \bar{w}, a_1)$ on C with $\bar{w}$ even (resp. odd). For C to be disk-like, we have $w(C) \in W(a_1) = 4\mathbb{Z}$. Thus the cost is $(n_0 + 3n_1)/4 = k$ for some integer $k \geq 1$.

    (b) The boundary only contains a_2 and a_4. Then $w(C) \equiv 2n_0 + n_1 \mod 2$, where n_0 (resp. n_1) is the total number of turns $(a_4, \bar{w}, a_2)$ on C with $\bar{w}$ even (resp. odd). For C to be disk-like, we need $w(C) \in W(a_2) = 6\mathbb{Z}$. Hence $2n_0 + n_1$ must be even and the cost $(2n_0 + n_1)/2 = k$ for some integer $k \geq 1$.

    (c) The boundary only contains a_3. Then $w(C) \equiv n \mod 3$, where n is the total number of turn on C. For C to be disk-like, we need $w(C) \in W(a_3) = 9\mathbb{Z}$. Hence n must be divisible by 3 and the cost $n/3 = k$ for some integer $k \geq 1$.

(2) Now suppose C is a disk-like piece containing at least one of the turns $(a_2, \bar{w}, a_1)$, $(a_4, \bar{w}, a_3)$, $(a_1, \bar{w}, a_4)$ or $(a_3, \bar{w}, a_2)$.

   (a) If C contains a_3 on the boundary, then it must have a turn $(a_3, \bar{w}, a_2)$ which already has cost $q_{3,\bar{w},2} = 1$.

   (b) If C does not contain a_3, then it includes 3 turns of the forms $(a_2, \bar{w}, a_1)$, $(a_1, \bar{w}', a_4)$ and $(a_4, \bar{w}'', a_2)$ respectively. In this case, the cost of C is at least

$$q_{2,\bar{w},1} + q_{1,\bar{w}',4} + q_{4,\bar{w}'',2} \geq \frac{1}{3} + \frac{1}{4} + \frac{1}{2} > 1.$$

In summary, any disk-like piece has cost at least 1. Let $t_{i,\bar{w},j}$ be the normalized number of turns $(a_i, \bar{w}, a_j)$ in a vector $x \in \mathcal{C}(c)$. Then the gluing conditions imply $t_{1,\bar{w},1} = t_{4,-\bar{w},2}$, $t_{1,\bar{w},4} = t_{3,-\bar{w},2}$ and $t_{2,\bar{w},1} = t_{4,-\bar{w},3}$. Therefore, the total cost

$$
\begin{aligned}
\sum_{i,\bar{w},j} q_{i,\bar{w},j} t_{i,\bar{w},j} &= \left(\frac{1}{4} + 1\right)\left(\sum_{\bar{w} \text{ even}} t_{1,\bar{w},1} + \sum_{\bar{w}} t_{1,\bar{w},4}\right) + \left(\frac{3}{4} + \frac{1}{2}\right)\sum_{\bar{w} \text{ odd}} t_{1,\bar{w},1} \\
&\quad + \frac{1}{3}\sum_{\bar{w}}(t_{3,\bar{w},3} + t_{4,\bar{w},3}) \\
&= \frac{5}{4}\sum_{\bar{w}}(t_{1,\bar{w},1} + t_{1,\bar{w},4}) + \frac{1}{3}\sum_{\bar{w}}(t_{3,\bar{w},3} + t_{4,\bar{w},3}).
\end{aligned}
$$

Combining with the normalizing condition, we have

$$\kappa_v(x) \leq \sum_{i,\bar{w},j} q_{i,\bar{w},j} t_{i,\bar{w},j} = \frac{5}{4} + \frac{1}{3},$$

by the duality method. Hence by Lemma 5.8, we have $\mathrm{scl}_{\mathrm{BS}(2,3)}([a, t^2]) \geq \frac{5}{24}$. □

If M and L are coprime, one can see that any reduced word of the form $g = a^{u_1} t a^{u_2} t \cdots a^{u_n} t a^{v_1} T a^{v_2} T \ldots a^{v_n} T$ can be rewritten as a reduced word $g = a^u t^n a^v T^n$, where $T = t^{-1}$. A trick using the Chinese remainder theorem shows $\mathrm{scl}_{\mathrm{BS}(M,L)}(g) = \mathrm{scl}_{\mathrm{BS}(M,L)}([a, t^n])$ for any such g. Thus it would be interesting to know how the sequence $\mathrm{scl}_{\mathrm{BS}(M,L)}([a, t^n])$ behaves as $n \to \infty$, where the complexity increases unboundedly. For example, does $\mathrm{scl}_{\mathrm{BS}(M,L)}([a, t^n])$ converges to some limit? If so, how fast does it converge?

One can lift $[a, t^n]$ to the infinite cyclic cover corresponding to $\ker h$, which is an infinite amalgam with presentation $\widetilde{G}(M, L) = \langle a_k, k \in \mathbb{Z} \mid a_k^M = a_{k+1}^L \rangle$. Then $g_n = a_0 a_n^{-1}$ is a lift of $[a, t^n]$ for each n. One can use techniques similar to the computations above to show that $\mathrm{scl}_{\langle \widetilde{G}(M,L), \langle a_0 \rangle \rangle}(g_n)$ converges *exponentially* in n to $1/2$. It is not clear whether a similar convergence holds for $\mathrm{scl}_{\mathrm{BS}(M,L)}([a, t^n])$. An exponential convergence seems unusual for scl of a family of words with linear word length growth, eg in free groups.

6.5 Convergence theorem

Throughout this subsection, let $F_2 = \langle a, t \rangle$ and consider $\mathrm{BS}(M, L)$ as the quotient by imposing the relation $a^M = ta^L t^{-1}$. We say a family of $\mathrm{BS}(M, L)$ is a *surgery family* if $d = \gcd(|M|, |L|) \to \infty$.

We observe from Remark 6.27 that

$$\mathrm{scl}_{\mathrm{BS}(M,L)}(atta^{-1}t^{-1} + t^{-1}) \to \mathrm{scl}_{F_2}(atta^{-1}t^{-1} + t^{-1}) = 1/2$$

as $|M|, |L| \to \infty$. In contrast, Proposition 6.25 shows that for certain chains like $c = at^2 + 2t^{-1}$, the convergence depends on how $|M|, |L|$ go to infinity, governed by how $d = \gcd(|M|, |L|)$ behaves. The purpose of this subsection is to study this convergence phenomenon for arbitrary chains in surgery families.

To describe the convergence in detail, we say a sequence $\{x_d\}$ is *quasirational* in d if there exists $\pi \in \mathbb{Z}_+$ and a rational function $\phi_p \in \mathbb{Q}(n)$ for each $0 \le p < \pi$ such that $x_{n\pi + p} = \phi_p(n)$.

Theorem 6.29 *For any chain $c \in B_1^H(F_2)$, let $\bar{c}$ be its image in $\mathrm{BS}(M, L)$. If $d = \gcd(M, L) \to \infty$, then $\mathrm{scl}_{\mathrm{BS}(M,L)}(\bar{c})$ converges to $\mathrm{scl}_{F_2}(c)$. Moreover, for M, L fixed, the sequence $\mathrm{scl}_{\mathrm{BS}(dM,dL)}(\bar{c})$ is eventually quasirational in d and converges to $\mathrm{scl}_{F_2}(c)$.*

Here F_2 is considered as the free HNN extension, that is, the graph of groups with a single vertex group $\mathbb{Z}$ and a single trivial edge group. Fix a rational chain $c = \sum r_i g_i \in B_1^H(F_2)$ with each g_i hyperbolic represented by a tight loop γ_i. We can apply Theorem 5.10 to compute $\mathrm{scl}_{F_2}(c)$, where disk-like pieces are the same as genuine disks since the edge group is trivial. Thus we notice that a piece in $\mathrm{BS}(M, L)$ becomes a disk-like piece in F_2 by ignoring winding numbers of turns if and only if the winding numbers of the arcs on the polygonal boundary sum to 0. We refer to such pieces as *stable* disk-like pieces.

When $d = \gcd(M, L)$ is sufficiently large, the cyclically reduced expression of g_i is also a cyclically reduced in $\mathrm{BS}(M, L)$. More precisely, with

$$N := \max_{a \in A_v} |w(a)|,$$

where A_v is the set of arcs obtained by cutting $\underline{\gamma} = \{\gamma_i\}$, each g_i is reduced when $d > N$. In particular, the image $\bar{g}_i$ of each g_i in $\mathrm{BS}(M, L)$ is hyperbolic with a constant complexity $\rho(\bar{g}_i)$ when $d > N$.

With such large d, recall from Lemma 5.8 that $\mathrm{scl}_{\mathrm{BS}(M,L)}(\bar{c})$ is computed as an optimization on the space $\mathcal{C}(\bar{c})$ where we essentially maximize κ_v. Let κ_v^* be the variant

of κ_v that counts the maximal number of *stable* disk-like pieces. Then the optimization in Lemma 5.8 with $\kappa_v(x)$ replaced by $\kappa_v^*(x)$ computes $\text{scl}_{F_2}(c)$ instead.

We prove the first assertion in Theorem 6.29.

Lemma 6.30 *With notation as above, $\text{scl}_{\text{BS}(M,L)}(\bar{c})$ converges to $\text{scl}_{F_2}(c)$ as $d \to \infty$.*

Proof By monotonicity of scl, it suffices to show for any $\epsilon > 0$, we have $\text{scl}_{\text{BS}(M,L)}(\bar{c}) \geq \text{scl}_{F_2}(c) - \epsilon$ for any d sufficiently large. Let S be any simple relative admissible surface for $\bar{c}$ in $\text{BS}(M,L)$. With the integer N defined above, note that each disk-like piece C of S is either stable or contains at least k arcs if $d \geq kN$. In fact, the sum of winding numbers of arcs on ∂C is divisible by d since both $w(C)$ (by equation (6–2)) and the contribution of each turn to $w(C)$ are divisible by d. Thus the sum is either 0, in which case C is stable, or has absolute value at least $d \geq kN$, where we must have at least k arcs.

Let $n > 0$ be the degree of S. Then the number of stable disk-like pieces in S is no more than $n\kappa_v^*(x)$ where $x = x(S)/n \in \mathcal{C}(c)$. By the observation above, the number of disk-like pieces in S that are not stable cannot exceed $(nN/d)\sum r_i A_i$, where A_i is the number of arcs that the edge space cut γ_i into. Since N and $\sum r_i A_i$ are fixed, for any d sufficiently large, the total number of disk-like pieces in S is no more than $n(\kappa_v^*(x) + \epsilon)$, from which the desired estimate follows. $\square$

Now we fix coprime integers $m, \ell \neq 0$ and focus on $\text{BS}(dm, d\ell)$. To ease the notation, we assume $m, \ell > 0$. With $d > N$, each $\bar{g}_i$ is reduced, thus $D_v = dm^{\rho(\bar{c})}\ell^{\rho(\bar{c})}\mathbb{Z}$ is linear in d, and $D_e = m^{\rho(\bar{c})}\ell^{\rho(\bar{c})}\mathbb{Z}$ does not depend on d. Therefore, the spaces $\mathcal{C}_v$ and $\mathcal{C}(\bar{c})$ are both eventually independent of d. What does depend on d is the set of disk-like vectors and the function κ_v. Denote $\mathcal{D}(v)^{(d)} = \mathcal{D}(v)$ and $\kappa_v^{(d)} = \kappa_v$ to emphasize the dependence, which we investigate to prove the second assertion in Theorem 6.29.

Throughout this subsection, we use Setup 2, where the criterion for disk-like vectors is simpler. Recall from Subsection 6.1 that each vector $x \in \mathcal{C}_v$ assigns non-negative weights to edges in the graph Y. An integer vector $x \in \mathcal{C}_v$ is disk-like if and only if its winding number $w(x) \in D_v$ and $\text{supp}(x) \subset Y$ is connected.

Lemma 6.31 *The extremal rays of the polyhedral cone $\mathcal{C}_v$ correspond to embedded cycles in Y. For each $x \in \mathcal{C}_v$, its support $\text{supp}(x)$ only depends on the unique open face of $\mathcal{C}_v$ containing x.*

Proof For each $x \in \mathcal{C}_v$, the requirement $\partial x = 0$ implies that x corresponds to a positive weighted sum of finitely many cycles ω_i in Y. Moreover, we may assume each ω_i to be embedded since each cycle is a sum of embedded ones. Thus $x \neq 0$ is on a extremal ray if and only if x corresponds to $\alpha\omega$ for some $\alpha > 0$ and an embedded cycle ω. It follows that vectors in the same open face have the same support. $\square$

We define the support of an open face $\mathcal{F}$ of $\mathcal{C}_v$ to be $\text{supp}(x)$ for any $x \in \mathcal{F}$.

To see how the set of disk-like vectors depend on d, similar to the proof of [28, Theorem 4.11], we consider the following embedding

$$
\begin{aligned}
\mathcal{E}_d : \mathbb{R}^{T_v} &\to \mathbb{R}^{T_v} \times \mathbb{R} \\
x &\mapsto (x, w(x)/|D_v|)
\end{aligned}
$$

where $|D_v| = dm^{\rho(c)}\ell^{\rho(c)}$ and w is defined as in the equation (6–3) by fixing a lift $\tilde{w} \in \mathbb{Z}$ for each $\bar{w} \in W_e$.

Note that $w(x)$ is the sum of two parts. The first part is the (weighted) sum of winding numbers of arcs a_i in x, which is independent of d. The second part is the sum of winding numbers contributed by turns, which is linear in d since both $o_e(\tilde{w}) = dm\tilde{w}$ and $t_e(\tilde{w}) = d\ell\tilde{w}$ are. In particular, $w(x)$ is affine linear in d for fixed x.

Observe that $\mathcal{E}_d(x)$ is an integer point if and only if x itself is and $w(x) \in D_v$. Thus, for each open face $\mathcal{F}$ of $\mathcal{C}_v$ with connected support, $\mathcal{E}_d$ gives a bijection between the set of disk-like vectors in $\mathcal{F}$ and the set of integer points in $\mathcal{E}_d(\mathcal{F})$.

We are now in a position to apply the following result of Calegari–Walker [11, Corollary 3.7]. To state it, we need the following definition.

Definition 6.32 A finite subset $S(d)$ of $\mathbb{Z}^k$ depending on d is *QIQ* if there exists some positive integer π such that for each $0 \leq p < \pi$, the coordinates of elements in $S(n\pi + d)$ are integral polynomials in n.

Lemma 6.33 (Calegari–Walker [11]) *Let $V(d)$ be a cone with integral extremal vectors affine linear in d. Then the vertex set of the integer hull (open or closed) of $V(n\pi + a) - 0$ is QIQ.*

Lemma 6.34 *The vertex set of $\text{conv}(\mathcal{D}(v)^{(d)}) + \mathcal{C}_v$ is eventually QIQ.*

Proof Only consider large d so that $\mathcal{C}_v$ does not depend on d. Fix an arbitrary open face $\mathcal{F}$ of $\mathcal{C}_v$ with connected support. Let $\{\omega_i\}_{i \in I}$ be finitely many embedded cycles

in Y representing the extremal rays on the boundary of $\mathcal{F}$. Denote the corresponding integer points in $\mathcal{C}_v$ also by ω_i, $i \in I$. Then $\mathcal{E}_d(|D_v|\omega_i) = (|D_v|\omega_i, w(\omega_i))$ is an integer point depending affine linearly on d. Thus the vertex set of the convex hull of integer points in $\mathcal{E}_d(\mathcal{F})$ is QIQ by Lemma 6.33. As we noticed earlier, these integer points are exactly the image of disk-like vectors in $\mathcal{F}$ under $\mathcal{E}_d$. Taking projection to the first coordinate shows that the vertex set of $\mathrm{conv}(\mathcal{D}(v)^{(d)} \cap \mathcal{F})$ is QIQ.

Since each disk-like vector lies in one of the finitely many open faces of $\mathcal{C}_v$, we conclude that the vertex set of $\mathrm{conv}(\mathcal{D}(v)^{(d)})$ is also QIQ. The conclusion follows since each vertex of $\mathrm{conv}(\mathcal{D}(v)^{(d)}) + \mathcal{C}_v$ is a vertex of $\mathrm{conv}(\mathcal{D}(v)^{(d)})$. $\qquad\square$

Proof of Theorem 6.29 We have proved the first assertion in Lemma 6.30. Now adopt the notation above to prove the second assertion for $\mathrm{BS}(dm, d\ell)$.

For d sufficiently large, Lemma 6.34 and the proof of Lemma 5.7 imply that, there is some positive integer π so that for each $0 \le p < \pi$, there are finitely many linear functions $\{f_i^{(n)}\}_{i \in I_p}$ where coefficients are rational functions in n, such that $\kappa_v^{n\pi+p}(x) = \min_{i \in I_p} f_i^n(x)$. Then $\mathrm{scl}_{\mathrm{BS}(dm,d\ell)}(\bar{c})$ is eventually quasirational by the linear programming problem formulated in the proof of Lemma 5.8. $\qquad\square$

6.6   Implementation

A direct implementation of the method introduced in Section 5 to compute scl would probably result in an algorithm with run time doubly exponential on the word length (see [6, Subsection 4.5]). However, in the case of Baumslag–Solitar groups, we have an algorithm computing $\mathrm{scl}_{\mathrm{BS}(M,L)}(c)$ with run time polynomial in the length of c if the complexity $\rho(c)$ is fixed. This uses the idea developed by Walker [32] to efficiently compute scl in free products of cyclic groups.

We use Setup 2 throughout out this subsection. Note that any piece has a multiple that is disk-like since D_v is finite index in $G_v = \mathbb{Z}$. Thus we will restrict our attention to disk-like pieces only. To compute scl, it comes down to maximizing the number of disk-like pieces by finding the best combination. The idea is to cut disk-like pieces into small building blocks to reduced the complexity of enumeration.

We first transform winding numbers into lengths. For each piece C, we think of each arc (resp. turn) on the polygonal boundary as a segment consisting of k unit intervals, where $k \in \{1, \ldots, |D_v|\}$ is the unique number congruent mod $|D_v|$ to the contribution of this arc (resp. turn) to $w(C)$. Then the length of C, ie the total number of unit intervals on its boundary, is divisible by $|D_v|$ since C is disk-like.
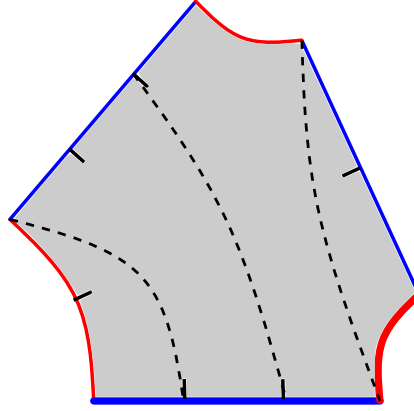
Figure 17: When $|D_v| = 3$, a piece of total length 12 can be cut into $k = 4$ blocks each of length $|D_v| = 3$, where the 4 chosen consecutive unit intervals are thickened. The two blocks in the middle each contains 2 genuine cuts, while the other two each contains 1 genuine cut.

It is easy to enumerate the possible types of unit intervals that appear in this way. For each arc $a_i \in A_v$, the number of intervals on it is the length of a_i, which does not exceed $|D_v|$. A similar bound holds for each turn from a_i to a_j, and there are no more than $|D_v|$ different turns given a_i, a_j. Note that $|A_v| \leq |c|$, where $|c|$ is the sum of word lengths of words involved in the chain c, so the total number N of types of unit intervals is bounded above by $2|c|^2|D_v|^2$.

Suppose a (disk-like) piece C has length $k|D_v|$ with $k \geq 1$. Choose k consecutive unit intervals and form $(k-1)$ cuts to divide C into k *blocks* so that each has length $|D_v|$ and each contains one of the k unit intervals. See Figure 17 for an illustration. Then each block B contains s cuts with $s \in \{0, 1, 2\}$. The case $s = 0$ appears only when the block itself is a disk-like piece of length $|D_v|$. In any case, let the orbifold Euler characteristic of a block B be $\chi_o(B) = 1 - s/2$. Then $\sum \chi_o(B) = 1$ where the summation is taken over the k blocks B obtained by cutting C.

For each block obtained from C, label the $|D_v|$ unit intervals on it cyclically by $\{1, \ldots, |D_v|\}$, where the one labeled by 1 is one of the k chosen intervals on C.

For the $|D_v|$ unit intervals on a block B, two nearby intervals are either separated by a cut or not. If there is no cut in between, we think of it as a *dummy cut*. Then each block consists of $|D_v|$ cuts, $s \in \{0, 1, 2\}$ of which are *genuine*. Moreover, one of the two intervals connected by a genuine cut must be labeled by 1. Each cut q contributes

to $\chi_o(B)$ by

$$\chi_o(q) := \begin{cases} 1/|D_v| & \text{if } q \text{ is dummy,} \\ 1/|D_v| - 1/2 & \text{if } q \text{ is genuine.} \end{cases}$$

Now we can form a linear programming problem to compute the maximum of $\kappa_v(x)$ over $x \in \mathcal{C}(c)$, from which we can obtain scl(c). Each variable x_q of the linear programming problem is the (normalized) number of a cut q (dummy or genuine) that possibly appear. The objective function to be maximized is $\sum_q x_q \chi_o(q)$, which is linear in x_q.

Here each cut q is encoded as a 5-tuple

$$(I_b, I_k, I_{k+1}, k, g),$$

where the binary variable $g = \textit{TRUE}$ if the cut is genuine, $k \in \{1, \ldots, |D_v|\}$ (indices taken mod $|D_v|$) indicates that the cut connects the k-th and $(k+1)$-th intervals on a block, and I_b, I_k, I_{k+1} represent the type of the intervals labeled by $1, k, k+1$ on the block respectively. This tuple satisfies the following restrictions:

(1) $g = \textit{FALSE}$ if $k \neq 1$ or $|D_v|$;

(2) I_k and I_{k+1} must be *consecutive* if $g = \textit{FALSE}$: If I_k is the last interval on an arc a_i, then I_{k+1} must be the first interval on a turn from a_i to some a_j, otherwise I_{k+1} must be the unique interval following I_k;

(3) $I_k = I_b$ if $k = 1$; and

(4) $I_{k+1} = I_b$ if $k = |D_v|$;

Besides the gluing and normalizing conditions, the linear programming problem here contains two more types of constraints.

(1) Boundary conditions that make sure cuts close up as blocks. This can be formulated to $\partial_B(x) = 0$ where ∂_B is defined as

$$\partial_B(I_b, I_k, I_{k+1}, k, g) := (I_b, I_{k+1}, k+1) - (I_b, I_k, k)$$

on basis elements and extends to a linear map to the abstract vector space with basis $\{(I_1, I_2, k) : 1 \leq k \leq |D_v|, I_1, I_2 \text{ are types of intervals}\}$.

(2) Gluing conditions that make sure genuine cuts glue up with each other. This can be formulated similarly as the gluing conditions for turns. We say two pairs (I, J) and (I', J') of types of intervals can be glued if I, J' are consecutive and I', J are consecutive. We say (I, J) and (I', J') are equivalent if they both can be glued with some (I'', J''). Then observe from the definition of being consecutive that if (I, J) and (I', J') are equivalent and (I, J) can be glued with (I'', J''), then

(I', J') can be glued with (I'', J'') as well. Thus it makes sense to say whether two equivalence classes can be glued.

Let V be the vector space spanned by equivalence classes of pairs. Let $\#_P$ and $\#_N$ be linear maps to V that vanish on $(I_b, I_k, I_{k+1}, k, g)$ unless $g = TRUE$. When $g = TRUE$, if $k = 1$, define $\#_P(I_b, I_k, I_{k+1}, k, g)$ to be the image of (I_k, I_{k+1}) and define $\#_N(I_b, I_k, I_{k+1}, k, g) := 0$; Otherwise $k = |D_v|$ and we define $\#_N(I_b, I_k, I_{k+1}, k, g)$ to be the image of (I_k, I_{k+1}) and define $\#_P(I_b, I_k, I_{k+1}, k, g) := 0$. Then the gluing conditions are that the $\overline{(I, J)}$ coordinate of $\#_P(x)$ equals the $\overline{(I', J')}$ coordinate of $\#_Q(x)$ whenever the equivalence classes $\overline{(I, J)}$ and $\overline{(I', J')}$ can be glued.

It is easy to show that the number of such tuples satisfying the restrictions is no more than $5|c|^4|D_v|^5 = 5|c|^4 d^5 |m|^{5\rho(c)}|\ell|^{5\rho(c)}$, which is polynomial in $|c|, d, m, \ell$ if the complexity $\rho(c)$ is bounded. This bounds the number of variables in the linear programming problem. Similar polynomial bounds on the number of constraints can be obtained. Fixing $\rho(c)$, this produces an algorithm that computes $\mathrm{scl}_{\mathrm{BS}(M,L)}(c)$ with time complexity polynomial in $|c|, |M|, |L|$ since linear programming problems can be solved by algorithms whose worst-case run time is polynomial [25]. Unfortunately, our algorithm does not run efficiently in practice, for example, $d^5|m|^{5\rho(c)}|\ell|^{5\rho(c)} \approx 6 \times 10^7$ is already terribly large when $M = 2$, $L = 3$, and the complexity $\rho(c) = 2$.

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