NOTES ON THE GROMOV BOUNDARY OF THE RAY GRAPH

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The goal of this appendix is to prove the following Proposition 1. This is a step in the proof of [BW18, Theorem 5.1.1]: Bavard and Walker start from a boundary point of the ray graph, choose a quasi-geodesic representative, pass to a cover-convergent subsequence, and then need to know that the limiting long ray is high-filling. The original proof of this fact appears to have a small gap, so we give a detailed proof.

We follow the notation and conventions of [BW18]. Thus S is the plane minus a Cantor set, with the isolated planar end denoted by ∞. The completed ray graph $\mathcal{R}$ has as vertices short rays, loops (based at ∞), and long rays, with edges given by disjointness. It has a main component $\mathcal{R}_0$, which contains all loops and all rays that are not high-filling. We know $\mathcal{R}_0$ is quasi-isometric to the ray graph, so they have the same Gromov boundary.

Also recall that cover-convergence is convergence in the conical cover: Each oriented loop or ray has a unique lift starting at the preferred lift $\widetilde{\infty}$ of ∞, and a sequence cover-converges when the endpoints of these preferred lifts converge in the boundary circle of the conical cover.

Proposition 1. *Let (x_i) be a quasi-geodesic representing a point p in the Gromov boundary of $\mathcal{R}_0$. Up to taking a subsequence, we suppose that (x_i) cover-converges to an element ℓ in the conical circle. Then ℓ is high-filling.*

About (infinite) unicorn paths. The proof relies on some geometric understanding of finite and infinite unicorn paths in the sense of [BW18, Section 3]. There are a few equivalent ways to define the unicorn path $P(a, b)$ between two oriented loops a and b. The path is simply a, b if they are disjoint, so we assume that a and b intersect and are in minimal position. There are finitely many intersections between a and b, and each intersection x uniquely determines a (not necessarily simple) loop that is made of an initial segment $(\infty x)_b$ of b and a terminal segment $(x\infty)_a$ of a. We only look at the simple ones, and order them (as we progress from a to b) so that the element has more portion on b and less portion on a. The key fact is that, given that we are looking for simple loops, taking more from b is equivalent to taking less from a. This already yields two equivalent definitions:

(1) List the intersections in the order along b, and collect the simple loops of the form above; or
(2) List the intersections in the order along a, and collect the simple loops of the form above.

The first description also leads to a third equivalent definition that inductively constructs the loops in the unicorn path as follows. Start with $a_0 = a$, and then for each $k \geq 0$, let x be the first intersection along b with a_k and let a_{k+1} be the concatenation of $(\infty x)_b$ and $(x\infty)_{a_k}$[1], which we set to be b and complete the process if a_k and b are already disjoint. It is easy to note by induction that:

(1) Perturbing $(\infty x)_b$ slightly in the direction of a_k at x gives a representative of a_{k+1} in minimal position with b,
(2) each a_k starts with a segment on b and then a segment on a.

From this, it should be clear why this is equivalent to the earlier definitions.

Now when b is a (long) ray, since the intersections with a are discrete along b, the first and third definitions still work out well, and they remain equivalent. This is the definition of the infinite

[1]The original definition in [BW18, Definition 3.2.1] appears to have a typo; this should be the correct version.

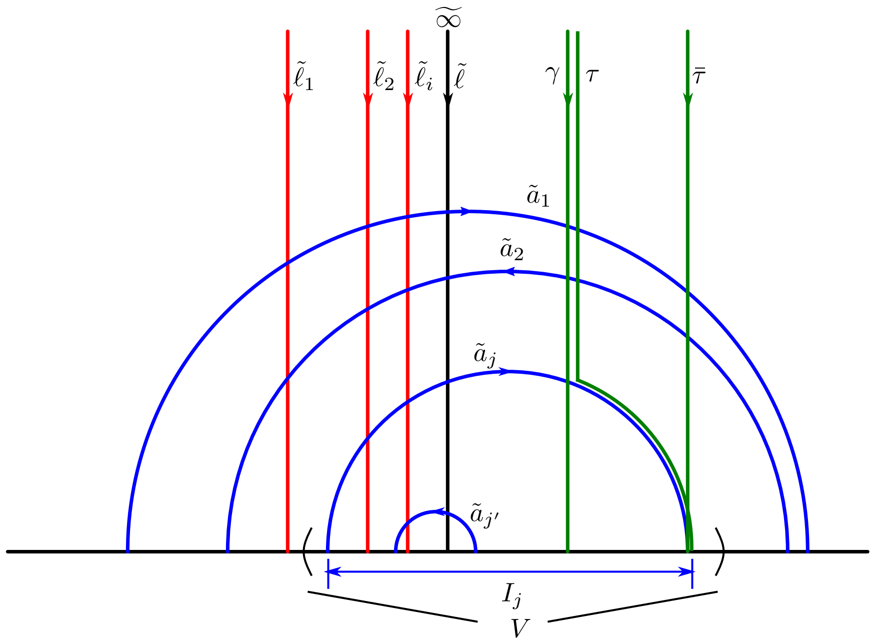

FIGURE 1. The lifts $\tilde{a}_j$'s of a shrink to the endpoint of $\tilde{\ell}$. The curve τ is the concatenation of γ and $\tilde{a}_j$ at their unique intersection x. Its straightening $\bar{\tau}$ is the lift of the geodesic representative of $(\infty x)_\gamma \cup (x\infty)_a$, and it lands in $I_j \subset V$.

unicorn path $P(a, b)$; the inductive description agrees with [BW18, Definition 3.3.1]. The following lemma can be observed from the first definition.

Lemma 2. *Let a be a loop. If a sequence of loops or rays ℓ_i cover-converges to a long ray ℓ, then for any n, there is $K > 0$ such that for all $i \geq K$, the first n elements in the unicorn path $P(a, \ell_i)$ agree with those in $P(a, \ell)$.*

Moreover, if a and ℓ have infinitely many intersections, then for any (two independent) sequences of integers $i_k, j_k \to \infty$, the j_k-th term in the unicorn path $P(a, \ell_{i_k})$ cover-converges to ℓ as $k \to \infty$.

Proof. The first assertion is obvious from the first definition, as ℓ_i starts almost the same as ℓ for all $i \geq K$ with K sufficiently large.

The second assertion follows from Figure 1 and the explanation below. Let $\widetilde{\infty}$ be the preferred lift of the planar end ∞. Let $\tilde{\ell}$ be the lift of ℓ that starts at $\widetilde{\infty}$. Let $\{\tilde{a}_i\}_{i=1}^{\infty}$ be the lifts of a based at lifts of intersections along $\tilde{\ell}$. These intersections are discrete since a is proper. Thus the lifts shrink down to the endpoint of $\tilde{\ell}$. So for any neighborhood V of ℓ in the conical circle, there is n such that for all $j \geq n$ the endpoints of $\tilde{a}_j$ determine a closed interval neighborhood $I_j \subset V$. For any ray or loop γ in I_j, realized as a geodesic from $\widetilde{\infty}$ to a point in I_j in the upper half plane model, there is a unique intersection x with $\tilde{a}_j$, and the concatenation $(\infty x)_\gamma \cup (x\infty)_a$ lies in I_j and thus in V. For any k large such that $j_k \geq n$ and i_k is large enough so that $\ell_{i_k} \in I_n$, the j_k-th term in the unicorn path is constructed using the j-th intersection along ℓ_{i_k} with a for some $j \geq j_k \geq n$ and hence its lift starting at $\widetilde{\infty}$ has endpoint in $I_n \subset V$. This completes the proof. □

Proof of the proposition. Now we give a proof of Proposition 1 by contradiction.

Proof of Proposition 1. Suppose ℓ is not high-filling, or equivalently by the characterization in [BW18, Lemma 3.5.4], the infinite unicorn path $P(a, \ell)$ is bounded in $\mathcal{R}_0$ for any oriented loop a.

Up to adding an element to the quasi-geodesic, we assume that x_0 is a loop and work with $a = x_0$. It is already shown in [BW18, Lemma 4.3.1] that ℓ is a loop-filling ray. In particular, ℓ must intersect

x_0 infinitely many times: If not, the unicorn path $P(x_0, \ell)$ is finite, and the second last term would give rise to a loop disjoint from ℓ.

Since (x_i) is a quasi-geodesic, by [BW18, Lemma 3.2.6], there is $C \in \mathbb{Z}_{\geq 0}$ (independent of k and n) such that for all $n > k$, the unicorn path $P(x_0, x_k)$ is contained in the C-neighborhood of $P(x_0, x_n)$. This is the only property in the proof that uses the fact that (x_i) is a quasi-geodesic, apart from the fact that $d(\alpha, x_i) \to \infty$ for any $\alpha \in \mathcal{R}_0$.

As we assumed ℓ to be not high-filling, the distance $d(\ell, x_k)$ is finite and goes to ∞ as $k \to \infty$. Fix any k such that $d(\ell, x_k) > C$. Then for any $n > k$, there is y_n on the unicorn path $P(x_0, x_n)$ with $d(x_k, y_n) \leq C$. We suppress the dependence on k as we will keep it fixed until the end of the argument. Now there is a path $z_{n,i}$ in $\mathcal{R}_0$ with $z_{n,1} = x_k$ and $z_{n,C} = y_n$. By taking a subsequence of n we may assume that each $z_{n,i}$ cover-converges to some $z_{\infty,i}$ as $n \to \infty$ fixing $1 \leq i \leq C$. In particular, y_n cover-converges to $y_\infty := z_{\infty,C}$. Disjointness is preserved under cover-convergence, thus these limits form a path in $\mathcal{R}_0$, and we have $d(x_k, y_\infty) \leq C$.

Suppose y_n is the i_n-th element on the unicorn path $P(x_0, x_n)$. If i_n is unbounded, then up to taking a subsequence, it goes to infinity. By the second assertion in Lemma 2, we must have that y_n cover-converges to $\ell = y_\infty$, which gives $d(x_k, \ell) \leq C$ and we get a contradiction.

Therefore, the sequence i_n must be bounded. Then by taking a subsequence, it is a constant i, and y_n is the i-th term on the unicorn path $P(x_0, \ell)$ for all n large by the first assertion in Lemma 2, as x_n cover-converges to ℓ. This shows that x_k is in the C-neighborhood of $P(x_0, \ell)$, which holds for all k large. However, our assumption implies that the path $P(x_0, \ell)$ is bounded and so is its C-neighborhood. This contradicts the fact that x_k escapes every bounded subset of $\mathcal{R}_0$. Thus ℓ must be high-filling as desired. □

References

[BW18] Juliette Bavard and Alden Walker. The Gromov boundary of the ray graph. *Trans. Amer. Math. Soc.*, 370(11):7647–7678, 2018.

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