STABLE COMMUTATOR LENGTH IN RIGHT-ANGLED ARTIN AND COXETER GROUPS

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Abstract. We establish a spectral gap for stable commutator length (scl) of integral chains in right-angled Artin groups (RAAGs). We show that this gap is *not* uniform, i.e. there are RAAGs and integral chains with scl arbitrarily close to zero. We determine the size of this gap up to a multiplicative constant in terms of the *opposite path length* of the defining graph. This result is in stark contrast with the known uniform gap 1/2 for elements in RAAGs. We prove an analogous result for right-angled Coxeter groups.

In a second part of this paper we relate certain integral chains in RAAGs to the *fractional stability number* of graphs. This has several consequences: Firstly, we show that every rational number $q \geq 1$ arises as the stable commutator length of an integral chain in some RAAG. Secondly, we show that computing scl of elements and chains in RAAGs is NP hard. Finally, we heuristically relate the distribution of scl for random elements in the free group to the distribution of fractional stability number in random graphs.

We prove all of our results in the general setting of graph products. In particular all above results hold verbatim for right-angled Coxeter groups.

1. Introduction

The stable commutator length is a relative version of the Gromov–Thurston norm. For a finite collection of loops $\gamma_1, \cdots, \gamma_k$ in a topological space X, its stable commutator length is the least complexity of surfaces bounding it, measured in terms of Euler characteristics (see Definition 2.1). This only depends on the fundamental group $G = \pi_1(X)$ and the conjugacy classes $g_1, \cdots, g_k$ representing the free homotopy classes of $\gamma_1, \cdots, \gamma_k$, and it is denoted as $\mathrm{scl}_G(g_1 + \cdots + g_k)$. We call this the stable commutator length of the (integral) chain $g_1 + \cdots + g_k$.

The stable commutator length arises naturally in geometry, topology and dynamics and has seen a vast development in recent years by Calegari and others [Cal09b, CF10, BBF16, Che20, HL20].

A group G has a *spectral gap* $C > 0$ *for elements* (resp. *chains*) if $\mathrm{scl}_G(g) \notin (0, C)$ for any element (resp. any chain) g in G. The largest such C is called the *optimal* spectral gap of G for elements (resp. *chains*). Various kinds of groups are known to have a gap for elements: word-hyperbolic groups [CF10], finite index subgroups of mapping class groups [BBF16], subgroups of right-angled Artin groups (defined below) [Heu19b], and 3-manifold groups [CH19]; see Theorem 2.17. The spectral gap property can be used to obstruct group homomorphisms since the stable commutator length is non-increasing under homomorphisms.

In contrast, much less is known about spectral gaps for chains. Calegari–Fujiwara [CF10] showed that hyperbolic groups have a spectral gap for chains. Their estimates have been made uniform and explicit in certain families of hyperbolic groups (Theorem 2.18). To our best knowledge, all previously known nontrivial examples with a spectral gap for chains are direct products of hyperbolic groups.

In this article we establish a *spectral gap* for chains in right-angled Artin groups. The *right-angled Artin group* $\mathrm{A}(\Gamma)$ associated to a simplicial graph Γ is the group with presentation

$$\mathrm{A}(\Gamma) = \langle \mathrm{V}(\Gamma) \mid [v, w]; (v, w) \in \mathrm{E}(\Gamma) \rangle,$$

which is not hyperbolic unless the graph contains no edge. Such groups are of importance due to their rich subgroup structure [Wis09, HW08, Ago13, Bri13, Bri17].

The gap is controlled by an invariant $\Delta(\Gamma) \geq 0$ of the defining simplicial graph Γ that we introduce, called *the opposite path length*; see Section 1.1.

Theorem A. *Let G be the right-angled Artin group associated to a simplicial graph Γ. Then the optimal spectral gap for integral chains in G is at least $\frac{1}{24+12\Delta(\Gamma)}$ and at most $\frac{1}{\Delta(\Gamma)}$.*

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The gap cannot be uniform among all right-angled Artin groups as there is an explicit finite graph Δ_m with $\Delta(\Delta_m) = m$ for any $m \in \mathbb{Z}_+$. The nonuniformness of the gap for chains is striking since right-angled Artin groups are known to have a uniform spectral gap 1/2 for elements [Heu19b, FFT19]. This is the first class of groups where the optimal gap for chains is known to be different from the optimal gap for elements. Using the nonuniformness, we construct countable groups where this difference becomes more apparent (Section 1.3).

We prove these results in the much more general setting of graph products (Theorem D). In particular, Theorem A holds verbatim for right-angled Coxeter groups, which are defined in the same way as right-angled Artin groups except that generators have order 2. For right-angled Coxeter groups, no gap was previously known in general, even for elements.

For a simplicial graph Γ we will construct a graph D_Γ and a chain c_Γ in $\mathrm{A}(D_\Gamma)$, called the *double chain of* Γ; see Definition 1.1. We will relate the stable commutator length of this chain linearly to the *fractional stability number* (Definition 1.2) of Γ (Theorem H). The latter invariant is well studied [SU11]. It is known that computing the fractional stability number is NP hard [GLS81] and that every rational number $q \geq 2$ is the fractional stability number of some graph [SU11, Proposition 3.2.2]. As consequences of this connection, we obtain the following two theorems.

Theorem B (NP-hardness, Theorem 7.14)**.** *Unless P=NP, there is no algorithm that, given a simplicial graph* Γ*, an element* $w \in \mathrm{A}(\Gamma)$ *and a rational number* $q \in \mathbb{Q}^+$*, decides if* $\mathrm{scl}_{\mathrm{A}(\Gamma)}(w) \leq q$ *with polynomial run time in* $|\mathrm{V}(\Gamma)| + |w|$*. The same holds for chains.*

This is in stark contrast to the case of free groups, as there is an algorithm by Calegari computing stable commutator length with polynomial run time in the word length of the input [Cal09b, CW09]; also compare to [Heu20].

Theorem C (Rational Realization, Theorem 7.13)**.** *For every rational* $q \in \mathbb{Q}_{\geq 1}$ *there is an integral chain* c *in a right-angled Artin group* $A(\Gamma)$ *such that* $\mathrm{scl}_{\mathrm{A}(\Gamma)}(c) = q$*.*

In the case of free groups, it is an unsolved conjecture of Calegari–Walker that the set of values of stable commutator lengths is dense in some intervals.

In the following subsections we will describe the generalization of our results to graph products, and collect some further results.

1.1. **Spectral Gaps for Integral Chains: Overview of the proof.** We now state the generalization of Theorem A to graph products and describe the main steps in its proof.

For a simplicial graph Γ, let $\{G_v\}_{v\in \mathrm{V}(\Gamma)}$ be a family of groups indexed by the vertex set $\mathrm{V}(\Gamma)$ of Γ. The *graph product* for this data is the free product $\star_{v\in\mathrm{V}(\Gamma)} G_v$ subject to the relations $[g_v, h_w]$ for every $g_v \in G_v$ and $h_w \in G_w$ whenever $(v, w) \in \mathrm{E}(\Gamma)$ is an edge of Γ. Graph products are generalizations of both right-angled Artin groups (which have vertex groups $\mathbb{Z}$) and right-angled Coxeter groups (which have vertex groups $\mathbb{Z}/2$).

For an integer $m \geq 1$, the *opposite path of length* m is the simplicial graph Δ_m with vertex set $\mathrm{V}(\Delta_m) = \{v_0, \ldots, v_m\}$ and edge set $\mathrm{E}(\Delta_m) = \{(v_i, v_j) \mid |i - j| \geq 2\}$. We define the *opposite path length* of a simplicial graph Γ to be

$$\Delta(\Gamma) := \max\{m \mid \Delta_m \text{ is an induced subgraph of } \Gamma\}.$$

Here a subgraph Λ of Γ is induced if any edge in Γ connecting $u, v \in \Lambda$ belongs to Λ.

Theorem D (Theorem 6.2)**.** *Let* Γ *be a simplicial graph, let* $\{G_v\}_{v\in\mathrm{V}(\Gamma)}$ *be a family of groups and let* $\mathcal{G}(\Gamma)$ *be the associated graph product. If* c *is an integral chain in* $\mathcal{G}(\Gamma)$ *then either* $\mathrm{scl}_{\mathcal{G}(\Gamma)}(c) \geq \frac{1}{12\Delta(\Gamma)+24}$ *or* c *is equivalent (see below) to a chain supported on the vertex groups, called* a vertex chain*. For vertex chains, there is an algorithm to compute* $\mathrm{scl}_{\mathcal{G}(\Gamma)}(c)$ *in terms of the stable commutator lengths in the vertex groups.*

Moreover, there is an integral chain δ *on* $\mathcal{G}(\Gamma)$ *such that*

$$\frac{1}{12(\Delta(\Gamma) + 2)} \leq \mathrm{scl}_{\mathcal{G}(\Gamma)}(\delta) \leq \frac{1}{\Delta(\Gamma)}.$$

The equivalence relation of chains, roughly speaking, is based on the following moves that does not change the stable commutator length. In an arbitrary group G with a chain c and elements $g, h \in G$ we have $\mathrm{scl}_G(c + g^n) = \mathrm{scl}_G(c + n \cdot g)$ for every $n \in \mathbb{Z}$ and $\mathrm{scl}_G(c + g) = \mathrm{scl}_G(c + hgh^{-1})$. If in addition g and h

commute, we have $\mathrm{scl}_G(c+g\cdot h)=\mathrm{scl}_G(c+g+h)$. We say that two chains c, c' in G are *equivalent*, if c can be transformed into c' by a finite sequence of these identities. See Definition 2.4 for the precise definition.

Formally, a *vertex chain* is of the form $c=\sum_v c_v$, where each c_v is a chain in the vertex group G_v. For right-angled Artin groups and right-angled Coxeter groups, any null-homologous vertex chain is equivalent to the zero chain and has zero scl. Thus Theorem A immediately follows from Theorem D. Moreover, we have a uniform gap 1/60 for all *hyperbolic* right-angled Coxeter groups; see Corollary 6.19.

In particular, Theorem D implies that groups with a gap for integral chains are preserved under taking graph products over finite graphs; see Corollary 6.4.

1.1.1. *Gaps for chains in graphs of groups.* The spectral gap result in Theorem D is based on a simple criterion for spectral gaps in graphs of groups that we prove. For simplicity, we state it for amalgamations.

Theorem E (Theorem 4.1, Long Pairings)**.** *Let $G=A\star_C B$ be an amalgamation and let $\sum_{i\in I} g_i$ be an integral chain. Then either*

$$\mathrm{scl}_G(c)\geq\frac{1}{12N}$$

or $c=\sum_{i\in I} g_i$ has a term $g=g_i$ such that $g^N=h^k h'd$ as reduced elements, where h is cyclically conjugate to the inverse of some term g_j in c, h' is a prefix (Definition 2.20) of h and $d\in C$.

We give two proofs of this criterion in Section 4, one using surfaces and the other using quasimorphisms.

To make use of this criterion, we reduce chains so that the exceptional algebraic relation $g^N=h^kh'd$ does not occur for a suitable N. In Section 5 we develop tools to achieve this goal for $N=D+2$, provided that all edge groups are *BCMS-D subgroups* (Definition 5.8). BCMS-D subgroups are generalizations of malnormal and central subgroups. In particular, malnormal subgroups are BCMS-1 and central subgroups are BCMS-0.

As key examples, for a graph product over a graph Γ, the subgroup corresponding to any induced subgraph is BCMS-D if the opposite path length $\Delta(\Gamma)=D$. Theorem D is obtained from the following estimate.

Theorem F (Theorem 5.1, BCMS gap)**.** *Let G be a graph of groups such that the embedding of every edge group $C\leq G$ is a BCMS-D subgroup. Then for any integral chain c in G, either c is equivalent to a chain supported on vertex groups, or*

$$\mathrm{scl}_G(c)\geq\frac{1}{12(D+2)}.$$

The special case where every edge group is malnormal in G is equivalent to that the fixed point set of each $g\neq id\in G$ has diameter at most 1 for the action on the Bass–Serre tree. In this case, we have the following corollary

Corollary G. *Let G be a graph of groups such that the embedding of each edge group $C\leq G$ is malnormal. Then for any integral chain c in G, either c is equivalent to a chain supported on vertex groups, or*

$$\mathrm{scl}_G(c)\geq\frac{1}{36}.$$

A similar result was obtained by Clay–Forester–Louwsma for hyperbolic elements in a group acting K-acylindrically on a simplicial tree; see [CFL16, Theorem 6.11].

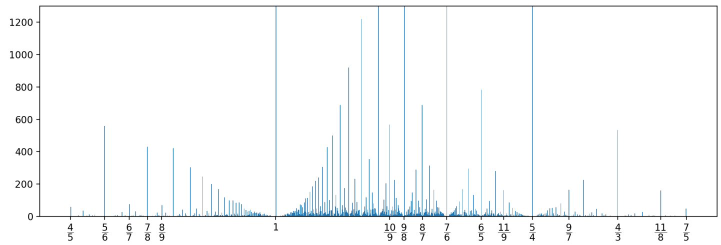
1.2. **fractional stability number and stable commutator length.** The algorithm mentioned in Theorem D computes stable commutator lengths of vertex chains as certain graph-theoretic quantities; see Section 7. As a special case, we discover a connection between stable commutator lengths of certain chains in right-angled Artin groups and the fractional stability numbers of graphs.

Definition 1.1 (Double Graphs and double chains)**.** For a simplicial graph Γ with vertices $\mathrm{V}(\Gamma)$ and edges $\mathrm{E}(\Gamma)$ we define the double graph D_Γ as the graph with vertex and edge set

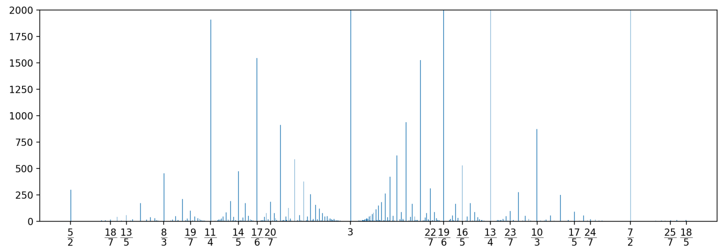
$$\begin{aligned}\mathrm{V}(D_\Gamma)&=\{\mathrm{a}_v,\mathrm{b}_v\mid v\in\mathrm{V}(\Gamma)\}\text{ and}\\ \mathrm{E}(D_\Gamma)&=\{(\mathrm{a}_v,\mathrm{a}_w),(\mathrm{a}_v,\mathrm{b}_w),(\mathrm{b}_v,\mathrm{a}_w),(\mathrm{b}_v,\mathrm{b}_w)\mid(v,w)\in\mathrm{E}(\Gamma)\}.\end{aligned}$$

Let d_Γ be the integral chain $\sum_{v\in\mathrm{V}(\Gamma)}[\mathrm{a}_v,\mathrm{b}_v]$. We call D_Γ the *double graph* and d_Γ the double chain in $\mathrm{A}(D_\Gamma)$.

A key feature of this construction is that $\mathrm{A}(D_\Gamma)$ is the graph product over the graph Γ with vertex groups $F(\mathrm{a}_v,\mathrm{b}_v)$.



(A) Histogram of $\mathrm{scl}_{F_2}(w)$ for 50.000 random words w uniformly choosen of length 24 in F_2



(B) Histogram of $\mathrm{scl}_{\mathrm{A}(D_\Gamma)}(d_\Gamma) = \frac{1}{2} \cdot \mathrm{fsn}(\Gamma)$ for 50.000 random graphs Γ uniformly choosen on 24 vertices. Here, d_Γ is the double chain in $\mathrm{A}(D_\Gamma)$ (Definition 1.1).

FIGURE 1. scl for random words in the free group vs. scl of random chains d_Γ in right-angled Artin groups $\mathrm{A}(D_\Gamma)$. In both cases, scl is rational and values with small denominator appear more frequent and the histogram exhibits a fractal behavior. In Section 7.3 we explain this distribution as the interference of (rounded) Gaussian distributions.

Definition 1.2 (fractional stability number)**.** A *stable measure* is a collection of non-negative weights $x = \{x_v\}_{v \in \mathrm{V}}$ assigned to vertices of Γ such that for any clique C (i.e. a complete subgraph) in Γ we have that $\sum_{c \in C} x_c \leq 1$. The *fractional stability number* of Γ is the supremum of $\sum_{v \in \mathrm{V}} x_v$ over all stable measures and denoted by $\mathrm{fsn}(\Gamma)$.

Theorem H (Theorem 7.11)**.** *Let Γ be a graph and let D_Γ and d_Γ be the associated double graph and double chain respectively. Then*

$$\mathrm{scl}_{\mathrm{A}(D_\Gamma)}(d_\Gamma) = \frac{1}{2} \cdot \mathrm{fsn}(\Gamma).$$

Combining with known results about fractional stability numbers, we deduce Theorems B and C. See Section 7 for the more general results about computations of stable commutator lengths of vertex chains in graph products.

The distributions of stable commutator length of random elements in free groups and fsn of random graphs are depicted in Figure 1. They exhibit a strikingly similar behavior: For both distributions values with small denominators appear more frequently, and the histograms exhibit some self-similarity. In Section 7.3 we analyze the distribution of scl and fsn further. This analysis allows us to describe a 5-parameter random variable X (Definition 7.15) which exhibits qualitatively the same distribution as scl and fsn. We use X to model both scl and fsn in Figure 7. While this is purely heuristic, it suggests that the distribution of scl and fsn converge to a similar distribution for large words or graph sizes; see Question 7.16.

1.3. **Groups with interesting scl spectrum.** The scl *spectrum* of a group is the range of the map $\mathrm{scl}_G : [G, G] \to \mathbb{R}_{\geq 0}$. The nonuniformness of spectral gap in Theorem A allows us to construct groups with interesting spectrum. There are few (classes of) groups where the spectrum of scl is fully known; see [Cal09a, Remark 5.20] and [Heu19a, Zhu08].

Theorem I. *There is a countable (right-angled Artin) group G such that $\mathrm{scl}_G(g) \geq 1/2$ for all $g \neq id \in [G, G]$ but there is no spectral gap for chains in G.*

Theorem J. *There is a countable group G such that $\mathrm{scl}_G(g) \geq 1/2$ for all $g \neq id \in [G,G]$, and its scl spectrum is dense in $[3/2, \infty)$.*

To the authors' best knowledge there was no group known that has a spectral gap for elements and the spectrum of elements becomes eventually dense, though free groups are conjectured to have this property.

These results are proved in Section 6.6.

1.4. **Organization.** This article is organized as follows. In Section 2 we recall basic results of stable commutator length, graph of groups and graph products respectively. In Section 4 we prove Theorem E estimating stable commutator length in graphs of groups. In Section 5 we will develop the theory of BCMS-D subgroups and prove Theorem F. Then we apply this to graph products of groups and prove Theorem D in Section 6. Finally in Section 7 we compute stable commutator lengths of vertex chains in graph products and relate them to fractional stability numbers.

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Contents

2. Background

We briefly introduce several concepts and set up some notations related to stable commutator length and graphs of groups. All results in this section are standard. Readers familiar with these topics may skip this section and refer to it when necessary.

2.1. **Stable Commutator Length.** We give the precise definition of the stable commutator length (scl) and recall some basic results. The reader may refer to [Cal09a, Chapter 2] for details.

Given a group G, let X be a topological space with fundamental group G. An integral chain is a finite formal sum of elements in G. Given an integral chain $\sum g_j$, consider loops γ_j in X so that the free homotopy class of γ_j represents the conjugacy class of g_j for each j.

An *admissible surface* is a pair (S, f), where S is a compact oriented surface and $f : S \to X$ is a continuous map such that the following diagram commutes and $\partial f_*[\partial S] = n(S,f)[\sqcup S^1_j]$ for some integer $n(S,f) > 0$, called the *degree* of the admissible surface.

$$\begin{array}{ccc} \partial S & \xrightarrow{i} & S \\ \partial f \downarrow & & \downarrow f \\ \sqcup S^1_j & \xrightarrow{\sqcup \gamma_j} & X \end{array}$$

Admissible surfaces exist if the chain is null-homologous, i.e. $\sum [g_j] = 0 \in H_1(G;\mathbb{Q})$. Let $\chi^-(S)$ be the Euler characteristic of S after removing disk and sphere components.

Definition 2.1. For any null-homologous integral chain $\sum g_j$ in G, we define

$$\mathrm{scl}_G(\sum g_j) := \inf_{(S,f)} \frac{-\chi^-(S)}{2 \cdot n(S,f)}.$$

When the chain represents a nontrivial rational homology class, we make the convention that $\mathrm{scl}_G(\sum g_j) = +\infty$.

We often omit the map f and refer to an admissible surface (S, f) simply as S.

In the special case where the chain is an element $g \in [G, G]$, this agrees with the algebraic definition using commutator lengths. See [Cal09a, Chapter 2] for more details as well as an algebraic definition for scl of integral chains.

Lemma 2.2. *For an integral chain $c = gh - g - h$ with $g, h \in G$, we have $\mathrm{scl}_G(c) \leq 1/2$.*

Proof. The fundamental group of a pair of pants S is the free group of rank 2, where the generators a, b can be chosen so that the boundary loops with the induced orientation are represented by ab, a^{-1} and b^{-1} respectively. The homomorphism $F_2 \to G$ determined by $a \mapsto g$ and $b \mapsto h$ corresponds to a map $f : S \to X$, which provides an admissible surface for c of degree one. Hence

$$\mathrm{scl}_G(c) \leq \frac{-\chi(S)}{2} = \frac{1}{2}.$$

□

Let $C_1(G)$ be the space of real 1-chains. By identifying g^{-1} with $-g$ in $C_1(G)$, scl is defined for any finite sum $\sum t_i g_i \in C_1(G)$, where $t_i \in \mathbb{Z}$. It is known that scl is linear on rays and satisfies the triangle inequality, and thus extends to a (semi-)norm on $C_1(G)$.

Proposition 2.3 (scl as a norm)**.** *Scl is a semi-norm on $C_1(G)$. In particular,* $\mathrm{scl}(c_1+c_2) \leq \mathrm{scl}(c_1)+\mathrm{scl}(c_2)$ *for any $c_1, c_2 \in C_1(G)$.*

Definition 2.4 (Equivalent chains)**.** Let $E(G)$ be the subspace of $C_1(G)$ spanned by elements of the following forms:

(1) $g^n - n \cdot g$, where $n \in \mathbb{Z}$ and $g \in G$,
(2) $hgh^{-1} - g$, where $g, h \in G$, and
(3) $gh - g - h$, where g and h are *commuting* elements in G.

We say two chains c and c' are *equivalent* if they differ by an element in $E(G)$.

Note that this is slightly different from the usual definition (e.g. [Cal09a, Definition 2.78]) by adding (3).

Proposition 2.5 (scl of equivalent chains)**.** *If c and c' are equivalent chains then*

$$\mathrm{scl}_G(c) = \mathrm{scl}_G(c').$$

Proof. Since scl is a semi-norm, this is to show that scl vanishes on each basis element of $E(G)$. For chains of the first two kinds, see [Cal09a, Section 2.6]. For a chain $gh - g - h$, where g and h commute, since $(gh)^n = g^n \cdot h^n$ for any $n \in \mathbb{Z}_+$, there is a thrice-punctured sphere with boundary components representing $(gh)^n$, g^{-n} and h^{-n} respectively. This gives rise to an admissible surface S for the chain $gh-g-h$ of degree n, which has $-\chi(S) = 1$. Letting n go to infinity, we have $\mathrm{scl}_G(gh - g - h) = 0$. □

We collect a few properties of stable commutator length. The main reference is [Cal09a].

Proposition 2.6 (Monotonicity and Retract)**.** *Let H, G be groups and let $f : H \to G$ be a homomorphism. Then for any chain c in $C_1(H)$ we have $\mathrm{scl}_H(c) \geq \mathrm{scl}_G(f(c))$. If in addition H is a retract of G, i.e. there is a homomorphism $r : G \to H$ such that $r \circ f = id_H$, then for any chain c in H we have that $\mathrm{scl}_H(c) = \mathrm{scl}_G(c)$.*

Proposition 2.7. *If $c = c_1 + c_2$ is a chain in $G = G_1 \star G_2$, where c_1 is supported on G_1 and c_2 is supported on G_2 then $\mathrm{scl}_G(c) = \mathrm{scl}_{G_1}(c_1) + \mathrm{scl}_{G_2}(c_2)$.*

Proof. This is a special case of [CH19, Theorem 6.2] since G is a graph of groups with vertex groups G_1, G_2 and a trivial edge group. □

Proposition 2.8 ([Cal09a, Theorem 2.101]). *Let G be a group and let $c = \sum_{i=1}^n g_i$ be a chain. Let $\tilde{G} = G \star \langle t_1 \rangle \star \cdots \star \langle t_{n-1} \rangle$ be the free product of G with $n-1$ infinite cyclic groups. Then*

$$\mathrm{scl}_G(c) = \mathrm{scl}_{\tilde{G}}(g_1 \cdot \prod_{i=1}^{n-1} t_i g_{i+1} t_i^{-1}) - \frac{n-1}{2}.$$

Proposition 2.9 (Index formula [Cal09a, Corollary 2.81]). *Let $H \trianglelefteq G$ be a finite index normal subgroup. The quotient $F = G/H$ acts on H by outer-automorphisms $h \mapsto f.h$, where $f.h$ is a well-defined conjugacy class in H. Then for any $h \in H$, we have*

$$\mathrm{scl}_G(h) = \frac{1}{|F|} \mathrm{scl}_H(\sum_{f \in F} f.h).$$

2.2. **Quasimorphisms.** Let G be a group. A map $\phi : G \to \mathbb{R}$ is called a *quasimorphism* if there is a constant $D > 0$ such that $|\phi(g) + \phi(h) - \phi(gh)| \leq D$ for all $g, h \in G$. The infimum of all such D is called the *defect* of ϕ and denoted by $D(\phi)$. Every bounded map and every homomorphism to $\mathbb{R}$ are trivially quasimorphisms but there are many nontrivial examples; see Example 2.15. A quasimorphism is called *homogeneous* if $\phi(g^n) = n \cdot \phi(g)$ for every $g \in G$ and $n \in \mathbb{Z}$. Every quasimorphism $\phi : G \to \mathbb{R}$ has a unique associated homogeneous quasimorphism $\bar{\phi}$ defined via

$$\bar{\phi}(g) := \lim_{n \to \infty} \frac{\phi(g^n)}{n}$$

which we call the *homogeneous representative* of ϕ.

Proposition 2.10 (Homogeneous Representative, [Cal09a, Lemma 2.58]). *Let $\phi : G \to \mathbb{R}$ be a quasimorphism with defect $D(\phi)$. Then the homogeneous representative $\bar{\phi}$ is in bounded distance to ϕ and satisfies $D(\bar{\phi}) \leq 2D(\phi)$.*

Here two quasimorphisms $\phi, \psi : G \to \mathbb{R}$ are in bounded distance if $\phi - \psi$ is bounded in the supremum norm.

Quasimorphisms are intimately connected to scl through Bavard's duality:

Theorem 2.11 (Bavard's Duality Theorem [Bav91], [Cal09a, Theorem 2.79]). *For any chain $c = \sum_{i \in I} n_i g_i$ with real coefficients $n_i \in \mathbb{R}$ we have*

$$\mathrm{scl}_G(c) = \sup_{\phi} \frac{\sum_{i \in I} n_i \phi(g_i)}{2D(\phi)},$$

where the supremum is taken over all homogeneous quasimorphisms $\phi : G \to \mathbb{R}$. Moreover, this supremum is achieved.

One can actually choose the homogeneous quasimorphism achieving the supremum in Bavard's duality to be the homogenization of a quasimorphism with nice properties. A quasimorphism ϕ is called *antisymmetric* if $\phi(g) = -\phi(g^{-1})$ for all $g \in G$.

Proposition 2.12 (Extremal Quasimorphisms). *Let G be a group. For any chain c in G there is a quasimorphism $\phi : G \to \mathbb{R}$ with $D(\phi) = 1/4$ that achieves the supremum of Bavard's duality, i.e. such that*

$$\mathrm{scl}_G(c) = \bar{\phi}(c)$$

where $\bar{\phi}$ is the homogenization of ϕ. Moreover, we may choose ϕ to be antisymmetric.

Proof. The statement without the moreover part is well known, and follows from the proof of [Cal09a, Theorem 2.70]. Now suppose ψ is such a quasimorphism with $D(\psi) = 1/4$ and $\bar{\psi}(c) = \mathrm{scl}_G(c)$. Let $\phi(g) := (\psi(g) - \psi(g^{-1}))/2$. Then ϕ is an antisymmetric quasimorphism with $D(\phi) \leq D(\psi) = 1/4$. It also follows by definition that $\bar{\phi} = \bar{\psi}$, and in particular $\bar{\phi}(c) = \bar{\psi}(c) = \mathrm{scl}_G(c)$. Thus by Barvard's duality, we must also have $D(\phi) \geq 1/4$ and hence $D(\phi) = 1/4$. This gives us the desired quasimorphism ϕ. ☐

Lemma 2.13. *For any homogeneous quasimorphism ϕ on G, we have $\phi(gh) = \phi(g) + \phi(h)$ if g and h commute.*

Proof. Note that for any $n \in \mathbb{Z}_+$ we have $(gh)^n = g^n h^n$ and

$$|\phi(gh) - \phi(g) - \phi(h)| = \frac{1}{n}|\phi(g^n h^n) - \phi(g^n) - \phi(h^n)| \le D(\phi)/n.$$

Taking $n \to \infty$ we have $\phi(gh) = \phi(g) + \phi(h)$. ☐

Proposition 2.14. *Let c be a chain in $G \cong G_1 \times G_2$. Then c is equivalent to a chain $c_1 + c_2$ where c_1 is supported on G_1 and c_2 is supported on G_2, and c_1, c_2 are integral chains if c is. Moreover,*

$$\mathrm{scl}_G(c) = \max\{\mathrm{scl}_{G_1}(c_1), \mathrm{scl}_{G_2}(c_2)\}.$$

Proof. Each element $g \in G$ can be written as $g_1 g_2$ for some $g_1 \in G_1$ and $g_2 \in G_2$, and thus g is equivalent to $g_1 + g_2$ as chains. The first claim easily follows from this.

Every homogeneous quasimorphism ϕ on G restricts to quasimorphisms ϕ_1 and ϕ_2 on G_1 and G_2 respectively. Then for the decomposition $g = g_1 g_2$ above for any $g \in G$, we have $\phi(g) = \phi(g_1) + \phi(g_2) = \phi_1(g_1) + \phi_2(g_2)$ by Lemma 2.13. It follows that $D(\phi) = D(\phi_1) + D(\phi_2)$ and $\phi(c) = \phi_1(c_1) + \phi_2(c_2)$ for the decomposition above.

Let ϕ be an extremal homogeneous quasimorphism for a chain c. For the decomposition $c = c_1 + c_2$ and $\phi = \phi_1 + \phi_2$, we have

$$\mathrm{scl}_G(c) = \frac{\phi(c_1 + c_2)}{2D(\phi)} \le \frac{|\phi_1(c_1)| + |\phi_2(c_2)|}{D(\phi_1) + D(\phi_2)} \le \max\left\{\frac{|\phi_1(c_1)|}{D(\phi_1)}, \frac{|\phi_2(c_2)|}{D(\phi_2)}\right\} \le \max\{\mathrm{scl}_{G_1}(c_1), \mathrm{scl}_{G_2}(c_2)\}$$

by Bavard's duality. This proves the second claim since the other direction $\mathrm{scl}_G(c_1 + c_2) \ge \mathrm{scl}_{G_i}(c_i)$ follows by the monotonicity of scl under the projection $G \to G_i$, where $i = 1, 2$. ☐

Example 2.15 (Brooks Quasimorphisms)*.* We describe a family of quasimorphisms on non-abelian free groups that certify a spectral gap in free groups. Let $F(\mathcal{S})$ be the free group on a generating set $\mathcal{S}$ and let $w \in F(\mathcal{S})$ be a reduced word. For an element $x \in F(\mathcal{S})$, let $\nu_w(x)$ be the maximal number of times that w is a subword of x i.e. the maximal n such that $x = x_0 w x_1 \cdots w x_n$, where $x_0, \ldots, x_n \in F(\mathcal{S})$ and this expression is reduced. We define $\phi_w : F(\mathcal{S}) \to \mathbb{Z}$ via $\phi_w : x \mapsto \nu_w(x) - \nu_{w^{-1}}(x)$. This map is called the *Brooks quasimorphism for w*. The family of these maps were introduced by Brooks in [Bro81] to show that the vector space of quasimorphisms is infinite dimensional. We will generalize Brooks quasimorphisms from free groups to amalgamated free products and HNN extensions in Section 4.1.

2.3. **Spectral Gaps in Stable Commutator Length.** We summarize some known methods and results on scl spectral gaps.

Definition 2.16. We say a group G has a *spectral gap* $C > 0$ for elements (resp. integral chains) if $\mathrm{scl}_G(c) \notin (0, C)$ for all elements (resp. integral chains) c in G.

The spectral gap property can be used to obstruct certain homomorphisms using monotonicity of scl (Proposition 2.6). A gap result for integral chains can also be used to estimate index of certain kinds of subgroups using the index formula (Proposition 2.9).

There are two main approaches to prove spectral gap results in a group G .

In light of Theorem 2.11 one approach is to construct for a given element g (resp. chain c) a homogeneous quasimorphism ϕ_g (resp. ϕ_c) of unit defect s.t. $\phi_g(g) \ge C$ (resp. $\phi_c(c) \ge C$) for a uniform $C > 0$. However, it is notoriously difficult to construct these maps which witness the optimal gap. For the free group, only two such constructions are available [Heu19b, CH19].

The other approach is to give a uniform lower bound of the complexity of all admissible surfaces. This is usually done by first simplifying admissible surfaces (sometimes in the language of disk diagrams) into certain normal form and then making use of a particular structure of the normal form; See for instance [DH91, Cul81, Che18, IK18, FST20, CH19].

Here we list some known spectral gap results for elements in Theorem 2.17 and for chains in Theorem 2.18. The list is by no means extensive.

Theorem 2.17.

(1) *(Calegari–Fujiwara* [CF10, Theorem A]*) Any δ-hyperbolic group with a generating set S has a spectral gap $C = C(|S|, \delta)$ for elements. Moreover, an element g has $\mathrm{scl}_G(g) = 0$ if and only if g^n is conjugate to g^{-n} for some $n \in \mathbb{Z}_+$.*

(2) *(Bestvina–Bromberg–Fujiwara* [BBF16, Theorem B]*) Let G be a finite index subgroup of the mapping class group* $\mathrm{Mod}(\Sigma)$ *of a possibly punctured closed orientable surface Σ. Then G has a spectral gap $C(G)$ for elements.*
(3) *(Chen–Heuer* [CH19, Theorem C]*) For any orientable 3-manifold M, its fundamental group has a spectral gap $C(M)$ for elements.*
(4) *(Heuer* [Heu19b, Theorem 7.3]*) Any (subgroup of a) RAAG has a spectral gap $1/2$ for elements. Moreover, any nontrivial element has positive scl. A new topological proof is given in* [CH19]*. Weaker results are obtained in* [FFT19] *and* [FST20].
(5) *(Clay–Forester–Louwsma* [CFL16, Theorem 6.9]*) Let $\{G_v\}$ be a family of groups with a uniform gap for elements. Then their free product also has a spectral gap for elements.*
(6) *(Chen–Heuer* [CH19, Theorem F]*) Let $\{G_v\}$ be a family of groups without 2-torsion such that they have a uniform gap for elements. Then their graph product also has a spectral gap for elements. The assumption on 2-torsion is unnecessary by our Theorem 6.2.*

Theorem 2.18.
(1) *(Calegari–Fujiwara* [CF10, Theorem A']*) Any δ-hyperbolic group with generating set S has a spectral gap $C = C(|S|, \delta)$ for integral chains. Moreover, an integral chain has zero scl if and only if it is equivalent to the zero chain. The following families of hyperbolic groups have uniform gaps even though the numbers of generators are unbounded.*
(2) *(Tao* [Tao16, Theorem 1.1]*) Any free group has a spectral gap $C = 1/8$ for integral chains.*
(3) *(Chen–Heuer* [CH19, Proposition 9.1]*) Free products of cyclic groups have a spectral gap $C = 1/12$ for integral chains. This is sharp for $\mathbb{Z}/2 \star \mathbb{Z}/3$.*
(4) *(Chen–Heuer* [CH19, Theorem 9.5]*) There is a uniform constant $C > 0$ such that the orbifold fundamental group of any closed hyperbolic 2-dimensional orbifold has a spectral gap C for integral chains.*

Note by Proposition 2.14 that groups with spectral gaps for chains is closed under direct products. Corollary 6.4 generalizes this to graph products. The authors are unaware of any groups that were previously known to have a spectral gap for chains other than direct products of hyperbolic groups.

2.4. **Amalgamated free products.** Let $G = A \star_C B$ be the amalgamated free product of groups A and B over a subgroup C. For any $g \in G \setminus C$, we may write

$$g = \mathrm{w}_1 \cdots \mathrm{w}_n \tag{2.1}$$

where $\mathrm{w}_i \in A \setminus C$ or $\mathrm{w}_i \in B \setminus C$ for all $i \in \{1, \dots, n\}$ such that the w_i's alternate between $A \setminus C$ and $B \setminus C$.

Remark 2.19. In an amalgamated free product $G = A \star_C B$ we use text font (e.g. a, b) to denote elements of $A \setminus C$ or $B \setminus C$. We refer to those elements as *vertex elements*. Ordinary roman letters (e.g. a, b) denote generic elements in G.

Definition 2.20 ((cyclically) reduced form for amalgamated free products)**.** We say that for an element $g \in G \setminus C$ the expression (2.1) is the *reduced form of g*. We define the *length of g* as n and denote it by $|g|$. Given the normal form (2.1) a *prefix* of g is an element $h \in G \setminus C$ with normal form $h = \mathrm{w}_1 \cdots \mathrm{w}_m$ where $0 \le m < n$.

If w_1 and w_n as in the reduced form (2.1) lie in different sets $A \setminus C$ and $B \setminus C$ then we say that g is *cyclically reduced*.

For $x_1, \dots, x_m \in G \setminus C$ we say that the expression $g = x_1 \cdots x_m$ is *a reduced decomposition of g* if there are reduced forms of each x_i such that their concatenation is a reduced form of g. Observe that g is cyclically reduced if and only if the expression $g \cdot g$ is reduced.

The reduced forms of an element are unique up to multiplication by C:

Proposition 2.21 (Reduced form for amalgamated free products [Ser03])**.** *Let $G = A \star_C B$ be an amalgamated free product and suppose that*

$$\mathrm{w}_1 \cdots \mathrm{w}_n = \mathrm{w}'_1 \cdots \mathrm{w}'_{n'}$$

where all w *terms alternate between $A \setminus C$ and $B \setminus C$. Then $n = n'$ and there are elements $d_0, \dots, d_n \in C$ with $d_0 = e = d_n$ such that $\mathrm{w}_i = d_{i-1} \mathrm{w}'_i d_i^{-1}$ for all $i \in \{1, \dots, n\}$.*

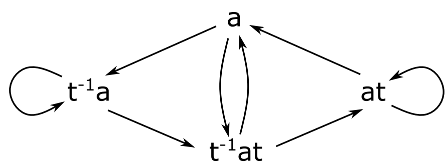


FIGURE 2. Possible concatenations of w_i

Corollary 2.22. *Let $G = A \star_C B$ be an amalgamated free product. Suppose that*

$$x_1 \cdots x_n = x'_1 \cdots x'_n$$

are two reduced decompositions such that $|x_i| = |x'_i|$ for all $i \in \{1, \dots, n\}$. Then there are elements $d_0, \dots, d_n \in C$ with $d_0 = e = d_n$ such that $x_i = d_{i-1} x'_i d_i^{-1}$ for all $i \in \{1, \dots, n\}$.

We will also need the following result later.

Proposition 2.23. *Let $g, h \in G$ be two elements. Then there are elements $y_1, y_2, y_3 \in G$ in reduced form and vertex elements (see Remark 2.19) $\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3 \in G$ such that*

$$\begin{aligned} g &= y_1^{-1} \mathrm{x}_1 y_2 \\ h &= y_2^{-1} \mathrm{x}_2 y_3 \\ (gh)^{-1} &= y_3^{-1} \mathrm{x}_3 y_1 \end{aligned}$$

as reduced decompositions, where possibly some y_i (resp. x_i) is the identity represented by the empty word (resp. letter).

Proof. Let $g = \mathrm{v}_1 \dots \mathrm{v}_m$ and $h = \mathrm{w}_1 \cdots \mathrm{w}_n$ be reduced. Let $0 \le i \le \min\{m, n\}$ be the largest integer such that $\mathrm{v}_{m-i} \cdots \mathrm{v}_m \mathrm{w}_1 \cdots \mathrm{w}_i = c \in C$.

Set $y_1^{-1} = \mathrm{v}_1 \cdots \mathrm{v}_{m-i-2}$, $y_2 = \mathrm{v}_{m-i} \cdots \mathrm{v}_m$ and $y_3 = \mathrm{w}_{i+2} \cdots \mathrm{w}_n$ and $\mathrm{x}_1 = \mathrm{v}_{m-i-1}$, $\mathrm{x}_2 = \mathrm{w}_{i+1}$ and $\mathrm{x}_3 = (\mathrm{v}_{m-i-1} c \mathrm{w}_{i+1})^{-1}$. By the minimality of i we see that $\mathrm{x}_3 \notin C$ unless $i = \min(m, n)$, in which case $\mathrm{x}_3 = id$ is represented by the empty word. Thus all of the expressions

$$\begin{aligned} g &= y_1^{-1} \mathrm{x}_1 y_2 \\ h &= y_2^{-1} \mathrm{x}_2 y_3 \\ (gh)^{-1} &= y_3^{-1} \mathrm{x}_3 y_1 \end{aligned}$$

are reduced. □

2.5. **HNN extensions.** Suppose C and C' are subgroups of a group A and $\phi : C \to C'$ is an isomorphism. Let $G = A\star_C$ be the associated HNN extension, obtained as the quotient of $A \star \langle t \rangle$ by relations $tct^{-1} = \phi(c)$ for all $c \in C$. For any $g \in G \setminus \{C, C'\}$, we may write

$$g = \mathrm{w}_1 \cdots \mathrm{w}_n \tag{2.2}$$

where

1. for each $i \in \{1, \dots, n\}$, w_i takes one of the following types:
   - $a \in A \setminus C$,
   - $t^{-1} a' t$ with $a' \in A \setminus C'$,
   - at with $a \in A$, or
   - $t^{-1} a$ with $a \in A$;

   We call such types *vertex elements* and denote them with text font e.g. a, b.
2. the possible types of any pair $(\mathrm{w}_i, \mathrm{w}_{i+1})$ are indicated by the oriented edges in Figure 2.

Note that for any $c_1, c_2 \in C$, the word $c_1 \mathrm{w}_i c_2$ can be rewritten into one of the same type as w_i, for instance, $c_1 \cdot t^{-1} a' t \cdot c_2 = t^{-1} a'' t$ with $a'' = \phi(c_1) a' \phi(c_2)$.

Definition 2.24 ((cyclically) reduced form for HNN extensions)**.** We say an expression as in (2.2) is a *reduced form* of g. Define the *length of* g to be n in (2.2), denoted as $|g|$. Given the reduced form (2.2) of g, a *prefix of* g is some $h = \mathrm{w}_1 \cdots \mathrm{w}_m$ with $0 \le m < n$. We say that g is *cyclically reduced* if the reduced expression $g = \mathrm{w}_1 \cdots \mathrm{w}_n$ satisfies in addition that $(\mathrm{w}_n, \mathrm{w}_1)$ is as in Figure 2 and we call such an expression a *cyclically reduced form.* We say that h is a cyclic conjugate of g if the reduced form of h is a cyclic permutation of the reduced form of g.

For a reduced element g, we say an expression $g = x_1 \cdots x_m$ is a *reduced decomposition* if there are reduced forms of each x_i so that the concatenation is a reduced form of g. Observe that g is cyclically reduced if and only if $g \cdot g$ is a reduced decomposition.

Reduced forms of a given element g is essentially unique:

Lemma 2.25 (Britton's lemma [LS77])**.** *Let $G = A\star_C$ be an HNN extension and suppose that*

$$\mathrm{w}_1 \cdots \mathrm{w}_n = \mathrm{w}'_1 \cdots \mathrm{w}'_{n'}$$

are two reduced forms of $g \in G \setminus C$. Then $n = n'$ and there are elements $d_0, \ldots, d_n \in C$ with $d_0 = e = d_n$ such that $\mathrm{w}_i = d_{i-1}\mathrm{w}'_i d_i^{-1}$ for all $i \in \{1, \ldots, n\}$.

From this we see that $|g|$ does not depend on the choice of reduced forms, and a reduced decomposition $g = x_1 \cdots x_m$ does not depend on the choice of reduced forms of x_i's.

Corollary 2.26. *Let $G = A\star_C$ be an HNN extension. Suppose that*

$$x_1 \cdots x_n = x'_1 \cdots x'_n$$

are two reduced decompositions such that $|x_i| = |x'_i|$ for all $i \in \{1, \ldots, n\}$. Then there are elements $d_0, \ldots, d_n \in C$ with $d_0 = e = d_n$ such that $x_i = d_{i-1}x'_i d_i^{-1}$ for all $i \in \{1, \ldots, n\}$.

Proposition 2.27. *Let $g, h \in G$ be two elements. Then there are elements $y_1, y_2, y_3 \in G$ and vertex elements $\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3 \in G$ such that*

$$\begin{aligned} g &= y_1^{-1}\mathrm{x}_1 y_2 \\ h &= y_2^{-1}\mathrm{x}_2 y_3 \\ (gh)^{-1} &= y_3^{-1}\mathrm{x}_3 y_1 \end{aligned}$$

as reduced expressions, where y_i and x_i might be the identity.

Proof. The proof of Proposition 2.23 works verbatim, interpreting vertex elements and reduced forms in the HNN extension context. □

2.6. **Graphs of Groups.** We briefly introduce graphs of groups to state the results of Sections 4 and 5 more compactly. Graph of groups is a generalization of both amalgamated free products and HNN extensions discussed in the previous sections. We refer to [Ser03] for details.

Let Γ be an oriented connected graph with vertex set V and edge set E. Each edge $e \in E$ is oriented with origin $o(e)$ and terminus $t(e)$. Denote the same edge with opposite orientation by $\bar{e}$, which provides an involution on E satisfying $t(\bar{e}) = o(e)$ and $o(\bar{e}) = t(e)$.

A *graph of groups* with underlying graph Γ is a collection of *vertex groups* $\{G_v\}_{v\in V}$, *edge groups* $\{G_e\}_{e\in E}$, and injections $t_e : G_e \hookrightarrow G_{t(e)}$, such that $G_e = G_{\bar{e}}$. Fix a pointed $K(G_v, 1)$ space X_v for each v and a pointed $K(G_e, 1)$ space X_e for each e. Each injection t_e determines a map $X_e \to X_{t(e)}$, based on which we can form a mapping cylinder $M_{e,t(e)}$, where we think of X_e and $X_{t(e)}$ as the subspaces on its boundary. Glue all such mapping cylinders along their boundary by identifying X_v in all $M_{e,v}$ (with $t(e) = v$) and identifying X_e with $X_{\bar{e}}$ in $M_{e,t(e)}$ and $M_{\bar{e},t(\bar{e})}$.

We refer to the resulting space X as the graph of spaces associated to the graph of groups, where the image of each X_e is called an edge space. The fundamental group $\pi_1(X)$ is called the *fundamental group of the graph of groups.* When there is no danger of ambiguity, we will simply refer to G as the graph of groups.

Theorem 2.28 ([Ser03])**.** *Every fundamental group of a graph of groups can be written as a sequence of amalgamated free products and HNN extensions over the edge groups.*

3. Graph products of groups

Graph products of groups generalize both right-angled Artin and right-angled Coxeter groups. They were introduced by Green in her thesis [Gre90]. We go through some basic concepts and then establish the pure factor decomposition and the centralizer theorem (Theorem 3.7). We will need these results in Sections 6 and 7.

Let Γ be a finite simplicial graph with vertex set $\mathrm{V}(\Gamma)$ and edge set $\mathrm{E}(\Gamma)$ and let $\{G_v\}_{v\in \mathrm{V}(\Gamma)}$ be a family of groups. Then, the *graph product* $\mathcal{G}(\Gamma, \{G_v\}_{v\in\mathrm{V}(\Gamma)})$ associated to this data is defined as the free product $\star_v G_v$ of the vertex groups subject to the relations $[g_v, g_w]$ for every $g_v \in G_v$, $g_w \in G_w$ with $(v, w) \in \mathrm{E}(\Gamma)$. If the family of groups G_v is understood we will simply denote the group as $\mathcal{G}(\Gamma)$.

When all $G_v = \mathbb{Z}$ (resp. $\mathbb{Z}/2$), we refer to $\mathcal{G}(\Gamma)$ as the right-angled Artin (resp. Coxeter) group, denoted as $\mathrm{A}(\Gamma)$ (resp. $\mathrm{C}(\Gamma)$).

A normal form of elements is developed in [Gre90]. Every element $g \in \mathcal{G}(\Gamma)$ can be written as a product $g_1 \cdots g_n$ where each g_i is in some vertex group. Following [Gre90, Definition 3.5] we say that n is the *syllable length* in such an expression. There are three types of moves on the set of words representing the same element:

- (syllable shuffling) if there is a subsequence $g_i \cdots g_j$ with $1 \le i < j \le n$ and $g_j \in G_{v_j}$ such that every g_k lies in a vertex group G_{v_k} and v_k is adjacent to v_j for all $i < k < j$, then we can replace it by $g_i g_j g_{i+1} \cdots g_{j-1}$, and similarly if every v_k is adjacent to v_i;
- (merging) if two consecutive letters g_i, g_{i+1} lie in the same vertex group G_v, we can merge them into a single letter $g_i g_{i+1} \in G_v$;
- (deleting) if some $g_i = 1$, then we can delete it.

Note that syllable shuffling preserves the syllable length while the other two moves reduce it.

We say that an expression $g_1 \cdots g_n$ is *reduced* if

- each g_i is nontrivial, and
- there is no subsequence $g_i \cdots g_j$ with $1 \le i < j \le n$ such that g_i, g_j lie in the same vertex group G_v, and every g_k lies in a vertex group G_{v_k} with v_k adjacent to v for all $i < k < j$.

Lemma 3.1 ([Gre90, Theorem 3.9])**.** *Every element $g \in \mathcal{G}(\Gamma)$ can be written as a reduced expression. This expression has minimal syllable length, and is unique up to syllable shufflings.*

The minimal syllable length of words representing g is denoted $|g|$, which is achieved by a word representing g if and only if the the word is reduced.

Similarly, a word is (proper) *cyclically reduced* if every cyclic permutation of its letters is reduced.

Lemma 3.2 (Proof of [Gre90, Theorem 3.24])**.** *Every conjugacy class in $\mathcal{G}(\Gamma)$ contains an element represented by a cyclically reduced word. Any two cyclically reduced words in the same conjugacy class differ by a cyclic permutation of the letters and syllable shuffling.*

Given a reduced expression $g = g_1 \cdots g_n$, its *support* is the induced subgraph consisting of vertices v such that some g_i lies in G_v. Since syllable shuffling does not change the support, by Lemma 3.1, the support does not depend on the choice of reduced expressions. We denote it by $\mathrm{supp}(g)$.

For an element $g \in \mathcal{G}(\Gamma)$ some conjugate $\bar{g} = p^{-1}gp$ is represented by a cyclically reduced word. By Lemma 3.2, the support $\mathrm{supp}(\bar{g})$ does not depend on the choice of p and we set $\Theta(g) := \mathrm{supp}(\bar{g})$.

The process of putting a word into a cyclically reduced word in the same conjugacy class does not enlarge the support (see the Proof of [Gre90, Theorem 3.24]), thus $\Theta(g)$ is the smallest support of elements in the conjugacy class of g.

Lemma 3.3. *For any $g \in \mathcal{G}(\Gamma)$, we have $\Theta(g) \subset \mathrm{supp}(g)$.*

Given two elements $g, h \in \mathcal{G}(\Gamma)$, we can relate the normal form of $g \cdot h$ to the reduced expressions of g, h as follows:

Proposition 3.4. *For any elements $g, h \in \mathcal{G}(\Gamma)$, there is a (possibly empty) clique $q = \{v_1, \dots, v_k\}$ for some $k \ge 0$ such that we may write $g = g_0 q_g x$ and $h = x^{-1} q_h h_0$ as reduced expressions with $q_g = g_1 \cdots g_k$ and $q_h = h_1 \cdots h_k$ with $g_i, h_i \in G_{v_i}$ and none of $g_i, h_i, g_i \cdot h_i$ is the identity for all $i \in \{1, \dots, k\}$ such that a reduced expression for gh is given by*

$$g \cdot h = g_0 \cdot q_{gh} \cdot h_0$$

where q_{gh} is given by $q_{gh} = (g_1 h_1) \cdots (g_k h_k)$.

Proof. Given $g, h \in \mathcal{G}(\Gamma)$ as in the proposition, choose x to be a word with the maximal syllable length such that $g = g'x$ and $h = x^{-1}h'$ are reduced expressions for some words g', h'. Given g' and h', choose q_g and q_h to be words with maximal syllable length such that the support of q_g and q_h is a clique $q = \{v_1, \dots, v_k\}$ and we can write $g' = g_0 q_g$, $h' = q_h h_0$ as reduced expressions for some words g_0, h_0. Define q_{gh} as in the proposition. By the maximality of x, none of the terms in q_g and q_h cancel and thus the support of q_{gh} is also equal to q. Note that for the expression

$$g_0 \cdot q_{gh} \cdot h_0,$$

$g_0 \cdot q_{gh}$ is reduced since $g' = g_0 \cdot q_g$ is reduced and has the same support. Similarly, $q_{gh} \cdot h_0$ is reduced. Finally, one cannot shuffle a letter in g_0 to merge with another in h_0 since this would contradict the maximality of g_q and h_q by Lemma 3.1. Thus $g_0 \cdot q_{gh} \cdot h_0$ is a reduced expression for $g \cdot h$. □

Next we introduce the pure factor decomposition. For a graph Γ, the *opposite graph* Γ^{opp} is the graph with the same vertices as Γ and where two vertices are adjacent if and only if they are not adjacent in Γ.

Let $C_1^*, \dots, C_{\ell^*}^*$ be the connected components of $\Theta(g)^{\text{opp}}$ each consisting of a single vertex and let $C_1, \dots, C_\ell$ be the connected components of $\Theta(g)^{\text{opp}}$ with more than one vertices. Letters of $\bar{g}$ in different components can be shuffled across. By combining letters in the same component via shuffling, we can write $\bar{g}$ as $\gamma_1^* \cdots \gamma_{\ell^*}^* \cdot g_1 \cdots g_\ell$ with $\text{supp}(\gamma_i^*) = C_i^*$ and $\text{supp}(g_i) = C_i$. Then it is easy to see that every g_i is cyclically reduced.

Now write $g_i = \gamma_i^{e_i}$ such that $e_i \in \mathbb{Z}_+$ and $\langle \gamma_i \rangle$ is maximal cyclic. Such an expression exists by [Bar07, Corollary 47]. We get

$$g = p \cdot \gamma_1^* \cdots \gamma_{\ell^*}^* \cdot \gamma_1^{e_1} \cdots \gamma_\ell^{e_\ell} \cdot p^{-1} \tag{3.1}$$

Definition 3.5 (Pure factor decomposition, pure factors, and primitive pure factors)**.** For any element g, an expression (3.1) is called a *pure factor decomposition* of g, where each γ_i^* and $g_i = \gamma_i^{e_i}$ is called a *pure factor* of g.

If $g = g^{e_1}$ with $e_1 = 1$ is its own pure factor decomposition and $|g| \geq 2$, then g is called a *primitive pure factor*.

Lemma 3.6. *Each pure factor of g is unique up to cyclic conjugation. The set of pure factors of g up to cyclic conjugation is uniquely determined by g.*

Proof. This directly follows from Lemma 3.2 and the fact that letters in different pure factors commute with each other. □

3.1. **Centralizers in Graph Products.** The goal of this subsection is to describe the centralizer of any element g in a graph product $\mathcal{G}(\Gamma)$.

Recall that $\Theta(g)$ is the support of any cyclically reduced representative of g, that $C_1^*, \dots, C_{\ell^*}^*$ denote the connected components of $\Theta(g)^{\text{opp}}$ that each consists of a single vertex and that $C_1, \dots, C_\ell$ denote the connected components of $\Theta(g)^{\text{opp}}$ containing more than one vertex. Finally let $D(g)$ be the subset of $\text{V}(\Gamma) \setminus \text{V}(\Theta(g))$ consisting of vertices which are adjacent to every vertex of $\Theta(g)$.

The following result fully characterizes the centralizer of an element in terms of the pure factors.

Theorem 3.7 (Centralizer Theorem)**.** *Let $g \in \mathcal{G}(\Gamma)$ be an element with pure factor decomposition*

$$g = p \cdot \gamma_1^* \cdots \gamma_{\ell^*}^* \cdot \gamma_1^{e_1} \cdots \gamma_\ell^{e_\ell} \cdot p^{-1},$$

where $\text{supp}(\gamma_i^*) = C_i^*$, $\text{supp}(\gamma_i) = C_i$ *and let $D(g)$ be defined as above. Then an element $h \in \mathcal{G}(\Gamma)$ commutes with g if and only if*

$$h = p \cdot \zeta_1^* \cdots \zeta_{\ell^*}^* \cdot \gamma_1^{f_1} \cdots \gamma_\ell^{f_\ell} \cdot z \cdot p^{-1},$$

where ζ_i^* lies in the centralizer $Z_{G_i^*}(\gamma_i^*)$, where G_i^* is the vertex group of C_i^*, $f_i \in \mathbb{Z}$, and $\text{supp}(z) \subset D(g)$.

This generalizes several similar results: In the case where g is itself a single pure factor, this is proved by Barkauskas [Bar07, Theorem 53]. In the case of right-angled Artin groups this has been done by Droms–Servatius–Servatius [SDS89] and in the case of graph products of abelian groups this has been done by Corredor–Gutierrez [CG12, Centralizer Theorem].

Lemma 3.8. *If g is cyclically reduced with pure factor decomposition*

$$g = \gamma_1^* \cdots \gamma_{\ell^*}^* \cdot \gamma_1^{e_1} \cdots \gamma_\ell^{e_\ell},$$

where $\text{supp}(\gamma_i^*) = C_i^*$, $\text{supp}(\gamma_i) = C_i$ *and the set $D(g)$ is defined as above. If $h \in \mathcal{G}(\Gamma)$ commutes with g and is supported on $\Theta(g) \cup D(g)$, then*

$$h = \zeta_1^* \cdots \zeta_{\ell^*}^* \cdot \gamma_1^{f_1} \cdots \gamma_\ell^{f_\ell} z,$$

where $\zeta_i^* \in Z_{G_i^*}(\gamma_i^*)$, where G_i^* is the vertex group of C_i^*, $f_i \in \mathbb{Z}$, and $\text{supp}(z) \subset D(g)$.

Proof. Recall that every vertex in $D(g)$ is adjacent to all vertices in $\Theta(g)$ and that letters supported on different components of $\Theta(g)^{\mathrm{opp}}$ commute with each other. So we can express h as a reduced expression $h = h_1^* \cdots h_{\ell^*}^* h_1 \cdots h_\ell z$, where each h_i^* (resp. h_i) is a reduced word with $\mathrm{supp}(h_i^*) \subset C_i^*$ (resp. $\mathrm{supp}(h_i) \subset C_i$), and z is a reduced word with $\mathrm{supp}(z) \subset D(g)$. Then

$$hgh^{-1} = \prod h_i^* \gamma_i^* (h_i^*)^{-1} \cdot \prod h_i \gamma_i^{e_i} h_i^{-1}.$$

Since different factors have disjoint support, we observe that $hgh^{-1} = g$ if and only if $h_i^* \gamma_i^* (h_i^*)^{-1} = \gamma_i^*$ and $h_i \gamma_i^{e_i} h_i^{-1} = \gamma_i^{e_i}$ for each i.

This reduces the problem to the case of a single pure factor. Hence by [Bar07, Theorem 53], we must have $h_i^* \in Z_{G_i^*}(\gamma_i^*)$ where G_i^* is the group associated to the vertex C_i^* and $h_i = \gamma_i^{f_i}$ for some $f_i \in \mathbb{Z}$. □

Lemma 3.9. *Suppose g and h are reduced words where the last letter h_v of h lies in G_v for some vertex $v \notin \mathrm{supp}(g)$ that is not adjacent to some $u \in \mathrm{supp}(g)$. Then g and h do not commute.*

Proof. Express $\mathcal{G}(\Gamma)$ as an amalgam $A \star_C B$, where $A = \mathcal{G}(\mathrm{St}(u))$, $B = \mathcal{G}(\Gamma \setminus \{u\})$, and $C = \mathcal{G}(\mathrm{Lk}(u))$. Here $\mathrm{St}(u)$ and $\mathrm{Lk}(u)$ denote the star and the link of u in Γ respectively. For a reduced expression $g = g_1 \cdots g_n$, we can pick out letters in G_u to obtain $g_1 \cdots g_n = b_0 g_{i_1} b_1 \cdots b_{s-1} g_{i_s} b_s$, where each $g_{i_k} \in G_u$ and each b_k is the product of letters outside G_u sitting in between g_{i_k} and $g_{i_{k+1}}$. Note that $b_k \in B \setminus C$ for all $k \neq 0, s$ since we start with a reduced expression of g. If $b_0 \in C$, then we can shuffle it across a_1. Thus we assume either $b_0 \in B \setminus C$ or $b_0 = id$. The same can be done for b_s, except for the case where $s = 1$ and both $b_0, b_s \in C$, in which we may assume one of them to be the identity.

In summary, this naturally expresses g as a reduced word $g = b_0 a_1 b_1 \cdots b_{s-1} a_s b_s$ in the amalgam $A \star_C B$, where $s \geq 1$ and $a_k = g_{i_k} \in G_u \subset A \setminus C$ and $b_k \in B \setminus C$ for each k, except that possibly $b_0, b_s = id$, or one of them is the identity and the other lies in C when $s = 1$.

Similarly we have $h = \beta_0 \alpha_1 \beta_1 \cdots \alpha_t \beta_t$ for some $t \geq 0$, where each $\alpha_i \in A \setminus C$ and $\beta_i \in B \setminus C$ except possibly $\beta_0 = id$. Note that we must have $\beta_t \in B \setminus C$ since h_v is the last letter of h as a reduced word in the graph product and $v \notin \mathrm{St}(u)$.

As words in the amalgam, we have

$$gh = b_0 \cdots a_1 \cdots a_s (b_s \beta_0) \alpha_1 \cdots \alpha_t \beta_t,$$
$$hg = \beta_0 \alpha_1 \cdots \alpha_t (\beta_t b_0) \cdots a_1 \cdots a_s b_s.$$

Since β_t as a reduced word in the graph product contains h_v and $v \notin \mathrm{supp}(g)$, while $\mathrm{supp}(b_0) \subset \mathrm{supp}(g)$, we know $\beta_t \cdot b_0 \in B \setminus C$. Thus hg is a reduced word in the amalgam except that possibly $\beta_0, b_s = id$.

If $gh = hg$, when written as reduced words in the amalgam they must have the same length and start and end on elements in the same factor groups (i.e. A and B). There are eight cases depending on whether $b_0, b_s, \beta_0 \in C$, but there are only two cases where gh and hg can be written as reduced words of the same type and length:

(1) $b_0 = \beta_0 = id$ and $b_s \notin C$, or
(2) $b_0, b_s, \beta_0 \notin C$, where $b_s \beta_0 \notin C$.

In both cases, hg ends with b_s and gh ends with β_t (or $b_s \beta_0$ when $t = 0$). If $gh = hg$, then we must have $\beta_t \in C b_s C$ (or $\beta_0 \in b_s^{-1} C b_s C$ when $t = 0$). Any element in $C b_s C$ (or $b_s^{-1} C b_s C$) as a word in the graph product is supported on $\mathrm{supp}(g) \cup \mathrm{St}(u)$, however β_t contains h_v and $v \notin \mathrm{supp}(g) \cup \mathrm{St}(u)$. This is a contradiction. Hence $gh \neq hg$. □

Lemma 3.10. *Suppose g is cyclically reduced. Let $D(g)$ be the set of vertices outside* $\mathrm{supp}(g)$ *and adjacent to all those in* $\mathrm{supp}(g) = \Theta(g)$ *as above. If $h \in \mathcal{G}(\Gamma)$ commutes with g, then h is supported on $\Theta(g) \cup D(g)$.*

Proof. Write h in a reduced expression. Denote $\mathrm{supp}(g) \cup D(g)$ by Δ and suppose $\mathrm{supp}(h) \not\subset \Delta$. Let h_v be the last letter in h with the property that $h_v \in G_v$ for some $v \notin \Delta$. Then h_v cuts h into a reduced expression $h_p h_v h_s$, where $\mathrm{supp}(h_s) \subset \Delta$. As vertices in $D(g)$ are adjacent to all vertices in $\Theta(g)$, by shuffling letters of h_s in $D(g)$ to the end, we may represent $h_s = h_s' h_z$ as a reduced word so that $\mathrm{supp}(h_s') \subset \Theta(g)$ and $\mathrm{supp}(h_z) \subset D(g)$.

As h_z commutes with g, we know $h' = h_p h_v h_s'$ also commutes with g, and h_v is also the last letter in h' supported outside Δ. Then the conjugate $h_s' h_p h_v$ must commute with $g' = h_s' g (h_s')^{-1}$. Note that $\mathrm{supp}(g') = \Theta(g)$ by Lemma 3.3 since we know $\mathrm{supp}(g') \subset \mathrm{supp}(g) = \Theta(g)$ as $\mathrm{supp}(h_s') \subset \Theta(g)$. Applying Lemma 3.9 to $h_s' h_p h_v$ and g' we get a contradiction. Thus we must have $\mathrm{supp}(h) \subset \Delta$. □

Now we prove Theorem 3.7.

Proof of Theorem 3.7. Since the centralizer of $p^{-1}gp$ is $p^{-1}Z_{\mathcal{G}(\Gamma)}(g)p$, it suffices to prove the theorem assuming $g = \bar{g}$ is cyclically reduced, i.e. $p = id$. Then by Lemma 3.10, any h commuting with g must be supported in $\Theta(g) \cup D(g)$. Thus the result follows from Lemma 3.8. □

Definition 3.11 (pure factor chain)**.** Suppose that $g \in \mathcal{G}(\Gamma)$ has an associated pure factor decomposition

$$g = p \cdot \gamma_1^* \cdots \gamma_{\ell^*}^* \cdot \gamma_1^{e_1} \cdots \gamma_\ell^{e_\ell} \cdot p^{-1}$$

where γ_i^* and γ_i and e_i are as in Equation (3.1). Then we define the associated *pure factor chain* g^{pf} *of* g initially as

$$g^{\mathrm{pf}} = \gamma_1^* + \cdots + \gamma_{\ell^*}^* + e_1\gamma_1 + \cdots e_l\gamma_l,$$

and then remove $e_i\gamma_i$ (resp. γ_i^*) if γ_i (resp. γ_i^*) is conjugate to its inverse.

For an integral chain $c = \sum_{i=1}^n c_i$ we define the associated pure factor chain c^{pf} as follows: Set $c^1 = \sum_{i=1}^n c_i^{\mathrm{pf}}$. If there is a term g_1^{-1} and $h_1 g_1 h_1^{-1}$ for some $g_1, h_1 \in \mathcal{G}(\Gamma)$ in c^1, define c^2 as the chain c^1 without g^{-1} and hgh^{-1}. If c^i is defined but still has terms g_i^{-1} and $h_i g_i h_i^{-1}$ for some $g_i, h_i \in \mathcal{G}(\Gamma)$, define c^{i+1} as c^i without g_i^{-1} and $h_i g_i h_i^{-1}$. Every such step reduces the number of terms by two, and thus, this process will eventually stop. We call the resulting chain the *pure factor chain* c^{pf} associated to c. Note that c^{pf} is equivalent to c.

By Lemma 3.6, the pure factor chains for different pure factor decompositions are equivalent in the sense of Definition 2.4.

Proposition 3.12. *Let c be an integral chain in $\mathcal{G}(\Gamma)$ equivalent (Definition 2.4) to a chain that consists of terms just supported on the vertex groups. Let c^{pf} be a pure factor chain. Then c^{pf} consists of terms which are just supported on vertex groups.*

Proof. Let $\gamma \in \mathcal{G}(\Gamma)$ be a primitive pure factor (Definition 3.5) so that γ is not conjugate to γ^{-1}. For any element $g \in \mathcal{G}(\Gamma)$ we define $\sigma_\gamma(g) = n$ if γ^n up to cyclic conjugation is a pure factor of g for some $n \in \mathbb{Z} \setminus \{0\}$. This is well defined by Lemma 3.6. The number n is uniquely determined since γ^n is cyclically reduced and has length $|n||\gamma|$, and γ is not conjugate to γ^{-1}. Set $\sigma_\gamma(g) = 0$ if no conjugate of γ^n for any n is a pure factor of g.

For a chain $c = \sum_{i \in I} \lambda_i c_i$ set $\sigma_\gamma(c) := \sum_{i \in I} \lambda_i \sigma_\gamma(c_i)$.

Claim 3.13. *If c and c' are equivalent chains (Definition 2.4). Then $\sigma_\gamma(c) = \sigma_\gamma(c')$.*

Proof. It suffices to show that $\sigma_\gamma(c) = 0$ for each basis element c in $E(G)$ as in Definition 2.4. Apparently $\sigma_\gamma(g) = \sigma_\gamma(pgp^{-1})$ since the pure factors of g up to cyclic conjugation only depends on the conjugacy class of g. The fact that $\sigma_\gamma(g^n) = n\sigma_\gamma(g)$ for all $n \in \mathbb{Z}$ follows from the definition.

It remains to show that $\sigma_\gamma(x_1) + \sigma_\gamma(x_2) = \sigma_\gamma(x_1 x_2)$ for two commuting elements $x, x_2 \in G$. If $\sigma_\gamma(x_1) = \sigma_\gamma(x_2) = \sigma_\gamma(x_1 \cdot x_2) = 0$ then the result trivially holds.

Without loss of generality assume that $\sigma_\gamma(x_1) \neq 0$. Let

$$x_1 = p \cdot \gamma_1^* \cdots \gamma_{\ell^*}^* \cdot \gamma_1^{e_1} \cdots \gamma_\ell^{e_\ell} \cdot p^{-1}$$

be the pure factor decomposition of x_1 with $\gamma_1 = \gamma$. Then x_2 has to be of the form

$$x_2 = p \cdot \zeta_1^* \cdots \zeta_{\ell^*}^* \cdot \gamma_1^{f_1} \cdots \gamma_\ell^{f_\ell} \cdot z \cdot p^{-1}$$

by Theorem 3.7. Note by the definition of $D(g)$ that $\mathrm{supp}(z)$ is disjoint from the support of any γ_i^* and γ_i. Thus z does not contribute to $\sigma_\gamma(x_2)$ and hence $\sigma_\gamma(x_2) = f_1$. For the same reason, we have $\sigma_\gamma(x_1 x_2) = e_1 + f_1$ from the expression

$$x_1 x_2 = p \cdot (\gamma_1^* \zeta_1^*) \cdots (\gamma_{\ell^*}^* \zeta_{\ell^*}^*) \cdot \gamma_1^{e_1+f_1} \cdots \gamma_\ell^{e_\ell+f_\ell} \cdot z \cdot p^{-1}.$$

Thus $\sigma_\gamma(x_1 x_2) = e_1 + f_1 = \sigma_\gamma(x_1) + \sigma_\gamma(x_2)$. This shows the claim. □

To conclude the proof of Proposition 3.12, Let c be an integral chain which is equivalent to a chain c' where every term is supported on a vertex. Let c^{pf} be a pure factor chain associated to c. If c^{pf} has a term γ which is not supported on vertices, then it is not conjugate to its inverse as such terms are removed in the beginning of the construction of c^{pf}. This term gives us a primitive pure factor γ such that $\sigma_\gamma(c^{\mathrm{pf}}) \neq 0$ since the number of terms in c^{pf} cannot be further reduced. On the other hand, we have $|\gamma| \geq 2$, since γ is not supported in a vertex. Thus $\sigma_\gamma(c') = 0$. This contradicts the above claim since c and c' are equivalent. □

We show in Corollary 6.18 that an element g in a RACG has $\mathrm{scl}(g) = 0$ if and only if g is equivalent to the zero chain. Jing Tao asked us if this can be characterized more explicitly in the following form. We confirm this explicit characterization.

Proposition 3.14. *For a RACG $\mathrm{C}(\Gamma)$, an element g is equivalent to the zero chain if and only if g is conjugate to g^{-1}. Moreover, this is equivalent to $g = ab$ with $a^2 = b^2 = id$.*

Proof. In any group, if $g = ab$ with $a^2 = b^2 = id$, then $g^{-1} = ba$ is conjugate to $g = ab$. It is also clear that if g is conjugate to g^{-1} then g is equivalent to the zero chain.

Let g be any element in a RACG $\mathrm{C}(\Gamma)$ with pure factorization

$$g = p \cdot \gamma_1^* \cdots \gamma_{\ell^*}^* \cdot \gamma_1^{e_1} \cdots \gamma_\ell^{e_\ell} \cdot p^{-1}.$$

As each γ_i^* necessarily has order two as it lies in a vertex group $\mathbb{Z}/2$, the pure factor chain $g^{\mathrm{pf}} = \sum e_i \gamma_i$, where the summation runs over i such that γ_i that is not conjugate to γ_i^{-1}. By Proposition 3.12, if g is equivalent to the zero chain, then g^{pf} is literally the zero chain, so γ_i is conjugate to γ_i^{-1} for all $1 \le i \le \ell$. As γ_i's and γ_i^*'s all commute, it follows that g and g^{-1} are conjugate.

Moreover, to see that $g = ab$ for some $a^2 = b^2 = id$, it suffices to show this for each γ_i due to the commutativity. Each γ_i is written as a cyclically reduced word w, and reversing the order of the letters gives a word $\bar{w}$ representing γ_i^{-1}, which must also be cyclically reduced. As these two cyclically reduced words represent the same conjugacy class since γ_i is conjugate to γ_i^{-1}, by Lemma 3.2, we know up to syllable shuffling w and $\bar{w}$, they differ by a cyclic permutation. That is, there is a reduced expression uv equivalent to w so that vu is equivalent to $\bar{w}$, where u, v are reduced words. It has the property that $uvvu = \gamma_i \cdot \gamma_i^{-1} = id$. By the following claim (with $n = 2$), we conclude that $u^2 = v^2 = id$ as desired.

Claim 3.15. *For any $n \ge 1$, suppose both $u_1 u_2 \cdots u_n$ and $u_n \cdots u_2 u_1$ are reduced expressions in $\mathrm{C}(\Gamma)$, where each u_i is a reduced word. If $u_1 u_2 \cdots u_n \cdot u_n \cdots u_2 u_1 = id$, then $u_i^2 = id$ for all i.*

Proof. We proceed by induction on the total length $\sum_i |u_i|$ of the word $u_1 \cdots u_n$. The result is immediate if the total length is 1. Suppose the result holds when the total length is at most $L - 1$ for $L \ge 2$, and consider such an expression with total length L. The expression $u_1 u_2 \cdots u_n \cdot u_n \cdots u_2 u_1$ must be reducible by assumption. Each u_i appears twice in the expression, we distinguish them by denoting the copy on the right as u_i' to avoid confusion.

As $u_1 u_2 \cdots u_n \cdot u_n' \cdots u_2' u_1'$ is the product of two reduced expressions, it must be the case that some letter g_v in some u_i can be shuffled all the way to merge with a letter g_v' in some u_j', where v is a vertex in Γ, and $1 \le i, j \le n$. We necessarily have $g_v = g_v'$ as the vertex group $G_v = \mathbb{Z}/2$.

We first show $i = j$. If $i < j$, then u_j sits in between u_i and u_j', so the g_v in u_i can be shuffled across u_j which also contains a copy of g_v, contradicting that $u_1 \cdots u_n$ is reduced. If $i > j$, then the g_v in u_j' can be shuffled across u_i' which contains a copy of g_v, contradicting that $u_n' \cdots u_1'$ is reduced.

Now given $i = j$, suppose $u_i = g_1 \cdots g_k$ as a reduced word, where g_i is the generator of the vertex group G_{v_i} of some vertex v_i. Then for some $1 \le s, t \le k$, the letter $g_s = g_v$ in u_i can be shuffled across $(g_{s+1} \cdots g_k) u_{i+1} \cdots u_n \cdot u_n' \cdots u_{i+1}' (g_1 \cdots g_{t-1})$ to cancel with $g_t = g_v'$ inside u_i', where $v = v_s = v_t$. We must have $s \ge t$ as otherwise g_s can be shuffled across $g_{s+1} \cdots g_{t-1}$ inside u_i to cancel g_t, contradicting that u_i is reduced.

First consider the case $s > t$. Then g_s can be shuffled to the end of u_i and g_t can be shuffled to the head of $u_i' = u_i$, so u_i is equivalent to $x v_i x$ as a reduced expression for some reduced word v_i, where $x = g_s = g_t$. It follows that

$$u_1 \cdots u_n = u_1 \cdots u_{i-1} x v_i x u_{i+1} \cdots u_n = u_1 \cdots u_{i-1} x v_i u_{i+1} \cdots u_n x$$

as words equivalent up to syllable shuffling. It follows that the subword $u_1 \cdots u_{i-1} x v_i u_{i+1} \cdots u_n$ of total length $L - 1$ is reduced, as part of the last reduced expression. Similarly,

$$u_n' \cdots u_1' = u_n' \cdots u_{i+1}' x v_i' x u_{i-1}' \cdots u_1' = x u_n' \cdots u_{i+1}' v_i' x u_{i-1}' \cdots u_1'$$

as equivalent reduced words, and $u_n' \cdots u_{i+1}' v_i' x u_{i-1}' \cdots u_1'$ is reduced, where $v_i' = v_i$. Hence by the induction hypothesis, as

$$\begin{aligned}(u_1 \cdots u_{i-1} x v_i u_{i+1} \cdots u_n)(u_n' \cdots u_{i+1}' v_i' x u_{i-1}' \cdots u_1') &= (u_1 \cdots u_{i-1} x v_i u_{i+1} \cdots u_n x)(x u_n' \cdots u_{i+1}' v_i' x u_{i-1}' \cdots u_1') \\ &= u_1 u_2 \cdots u_n \cdot u_n \cdots u_2 u_1 = id,\end{aligned}$$

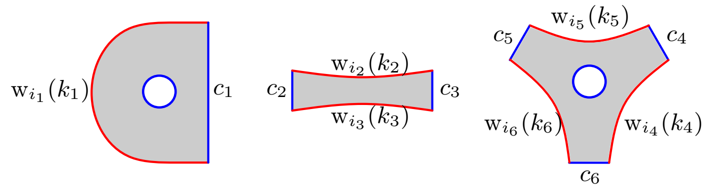


FIGURE 3. Pieces with or without a hole in the interior

where we think of x and v_i both as reduced words in the reduced expression, we must have $u_j^2 = id$ for all $j \neq i$ and $v_i^2 = id$, which implies $u_i^2 = xv_i^2x = x^2 = id$.

Now consider the remaining case $s = t$. In this case $g_s = g_t$ commutes with all remaining letters in u_i as well as those in u_j for $j > i$ by the same analysis as above. So we can write u_i equivalently as $v_i x$ and xv_i, which are reduced expressions where $x = g_s$ commutes with v_i. The by the same argument, $u_1 \cdots u_{i-1} v_i u_{i+1} \cdots u_n$ is a reduced word of total length $L-1$ as a subword of $u_1 \cdots u_{i-1} v_i u_{i+1} \cdots u_n x$ and similarly $u'_n \cdots u'_{i+1} v_i u_{i-1} \cdots u_1$ is reduced. As the product of them is the identity, the induction hypothesis implies $u_j^2 = id$ for all $j \neq i$ and $v_i^2 = id$, which implies $u_i^2 = v_i^2 x^2 = x^2 = id$. □

□

4. GAPS FROM SHORT OVERLAPS

Let G be a group splitting over a subgroup C, that is, G is either an amalgam $A \star_C B$ or an HNN extension $A\star_C$. In either case, G is a graph of groups with a unique edge group C, realized as a graph of spaces X with a single edge space.

Consider an integral chain $d = \sum g(i)$, where each $g(i) = \mathrm{w}_1(i) \cdots \mathrm{w}_{L_i}(i)$ is a cyclically reduced word and does not lie in the vertex groups. For any integral chain $d + d'$, where d' is a sum of elements in vertex groups, any admissible surface S of degree n for $d + d'$ can be considered as an admissible surface for d of the same degree with extra boundary components representing curves in vertex groups. This is called an admissible surface for d *relative to the vertex groups*.

Then S can be simplified into the *simple normal form* in the sense of [Che20, Section 3.2], which does not increase $-\chi^-(S)$ and does not change the degree. This means that S is obtained by gluing *pieces* together, where each piece is a polygon possibly containing a hole in the interior, with $2k$ sides alternating between *arcs* and *turns* for some $k \in \mathbb{Z}_+$; see Figure 3. Topologically, each piece is either a disk or an annulus. Turns are places that these pieces glue along, and arcs are part of ∂S. They carry labels that we describe as follows.

In the case of an amalgam, each piece is either supported in A or B. If a piece is supported in A, then each arc is labeled by some $\mathrm{w}_i(k) \in A \setminus C$, and each turn is labeled by some element $c \in C$, which we refer to as the winding number of the turn. The product of labels on the polygonal boundary of each piece supported in A (resp. B) defines a conjugacy class in A (resp. B), which is id if and only if the piece is a disk (i.e. has no hole inside).

In the case of an HNN extension, each piece is supported in the vertex group A. Each arc is labeled by some $\mathrm{w}_i(k) \in A \setminus C$, and each turn is labeled by some element $c \in C$, the winding number of the turn. Recall from Section 2.5 that each $\mathrm{w}_i(k)$ falls into one of four types. If a turn travels from some $\mathrm{w}_i(k)$ to $\mathrm{w}_j(\ell)$, then the possible types of $(\mathrm{w}_i(k), \mathrm{w}_j(\ell))$ are

$$(at \text{ or } a, t^{-1}a \text{ or } a) \quad \text{and} \quad (t^{-1}a \text{ or } t^{-1}at, at \text{ or } t^{-1}at).$$

It follows that the product of labels on the polygonal boundary defines a conjugacy class in A. The conjugacy class is id if and only if the piece is a disk.

In both cases, each disk piece has at least two turns since each $\mathrm{w}_i(k) \notin C$.

Pieces are glued together along *paired turns* to form S. Here a turn from $\mathrm{w}_i(k)$ to $\mathrm{w}_j(\ell)$ with winding number $c \in C$ is uniquely paired with a turn from $\mathrm{w}_{j-1}(\ell)$ to $\mathrm{w}_{i+1}(k)$ with winding number c^{-1}. The gluing guarantees that each boundary component of S is labeled by a conjugate of $g(i)^k$ for some $k \in \mathbb{Z}_+$. The way we glue pieces together is encoded by the *gluing graph* Γ_S, where each vertex corresponds to a piece and each edge corresponds to a gluing of two paired turns. For each vertex v, let $d(v)$ be its valence in Γ_S, and let $\delta(v) = 1$ if the corresponding piece is a disk and $\delta(v) = 0$ otherwise (i.e. for an annulus piece).

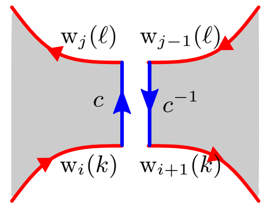


FIGURE 4. Two paired turns

If we cap off the hole in each annulus piece in S, then the surface deformation retracts to the graph Γ_S. Recall that the Euler characteristic $\chi(\Gamma_S)$ can be computed as $\sum_v[1-d(v)/2]$, so we have

$$-\chi(S) = -\chi(\Gamma_S) + |V_A| = \sum_v [d(v)/2 - \delta(v)], \tag{4.1}$$

where $|V_A|$ is the number of annuli pieces. Note that $d(v)/2-\delta(v)\geq 0$, and the equality holds if and only if v is a disk piece with two turns (i.e. v has valence 2 in Γ_S).

Theorem 4.1. *Suppose G is a group that splits over a subgroup C. Let $c=\sum_{i=1}^n g(i)$ be an integral chain in G where each term either lies in a vertex group or is cyclically reduced.*

Fix an integer $N\in\mathbb{Z}_+$. Then either

$$\mathrm{scl}_G(c) \geq \frac{1}{12N}$$

or for any cyclically reduced $g=g(i)$, $i\in\{1,\dots,n\}$, we have

$$g^N = h^k h' d,$$

where

- *h is a cyclically reduced word conjugate to $g(j)^{-1}$ for some $j\in\{1,\dots,n\}$, and $k\in\mathbb{Z}_{\geq 0}$,*
- *h' is a prefix of h and*
- *$d\in C$.*

Proof. Without loss of generality, assume $g(1)$ is cyclically reduced and no such equations hold for $g=g(1)$. We show $\mathrm{scl}_G(c)\geq \frac{1}{12N}$.

Start with any admissible surface S for c without sphere or disk components. For any large integer M, there is a finite normal cover $\tilde{S}$ of S where each component of $\partial\tilde{S}$ covers some component of ∂S with degree greater than M. In particular, this shows that, up to taking finite covers, any boundary component of S winding around $g(1)$ represents $g(1)^{qN+r}$ for some $q,r\in\mathbb{Z}_+$ where the remainder r is negligible compared to q. Thus in the following estimate, we will assume for simplicity that whenever a boundary component of S winds around $g(1)$, it actually winds around $g(1)$ some N-multiple of times.

Remove elements in c supported on vertex groups to obtain an integral chain c_0. Then as explained above, we can think of S as an admissible surface for c_0 relative to the vertex groups. Up to homotopy and compression, we can put S into the simple normal form, which does not affect the boundary; see [Che20, Lemma 3.7]. For each boundary component representing $g(1)^{qN}$, cut it into q segments, so that each segment is labeled by the cyclically reduced word representing $g(1)^N$. Each segment consists of $L_1\cdot N$ distinct arcs (some with same labels) and thus witnesses $L_1\cdot N$ pieces, some of which might be counted multiple times (since some arcs might lie on the same piece).

We claim that at least one of these pieces witnessed along a segment is represented by a vertex v in the gluing graph Γ_S such that $d(v)/2-\delta(v)>0$. If not, then each such a piece is a disk with two turns. Such rectangles glue to a long strip (see Figure 5), whose boundary shows that $g(1)^N c_1 w c_2 = id$ for some $c_1,c_2\in C$, where w is the word on the opposite side of $g(1)^N$ and must be a reduced subword of some $g(j)^m$ ($g(j)$ represents the loop that the boundary component on the opposite side of the strip maps onto). In algebraic terms, this implies an equation that should not exist by our assumption.

Therefore, for each segment σ as above, we can choose a piece $v(\sigma)$ witnessed by σ so that $d(v)/2-\delta(v)>0$. It is possible that $v(\sigma)=v(\sigma')$ for distinct segments σ,σ'. Thinking of such pieces as vertices on Γ_S, each $v=v(\sigma)$ either has $d(v)\geq 3$ or has $d(v)\leq 2$ and $\delta(v)=0$. In the former case, such a vertex is witnessed by at most $d(v)$ segments, and hence each segment witnessing v contributes at least $\frac{1}{d(v)}[d(v)/2-\delta(v)]\geq$

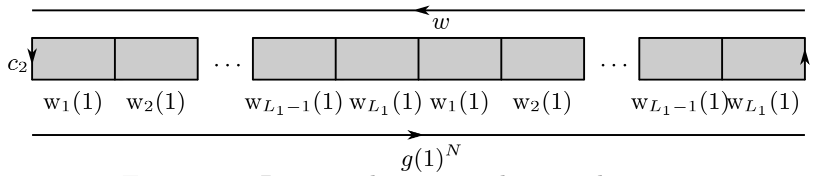


FIGURE 5. Rectangular pieces glue to a long strip.

$\frac{d(v)-2}{2d(v)} \geq 1/6$ to the right-hand side of equation (4.1). In the latter case, such a vertex has $\delta(v) = 0$, and hence each segment witnessing v contributes at least $\frac{1}{d(v)}[d(v)/2 - \delta(v)] = 1/2$ to the right-hand side of (4.1). Thus in any case, each segment contributes at least 1/6 to $-\delta(S)$, and the total number of such segments is n/N, where n is the degree of S.

Hence we obtain

$$\frac{-\chi(S)}{2n} \geq \frac{1}{6} \cdot \frac{n}{N} \cdot \frac{1}{2n} = \frac{1}{12N}.$$

Since S is arbitrary, this gives the desired estimate.

□

4.1. **Proof of Theorem 4.1 using quasimorphisms.** In this section we will give an alternative proof to Theorem 4.1 using explicit quasimorphisms. The quasimorphisms will be similar to the counting quasimorphisms discovered by Brooks [Bro81]; see also Example 2.15. For amalgamated free products, this is also similar to [CFL16].

Let G be an amalgamated free product or HNN extension which splits over a group C. Let $w \in G$ be a cyclically reduced element. Then we define $\nu_w : G \to \mathbb{N}$ as follows. For any $g \in G$ let $\nu_w(g)$ be the largest integer n such that g has reduced decomposition

$$g = g_0 w_1 g_1 \cdots w_n g_n,$$

where $g_i \in G$ is possibly the empty word and $w_i \in CwC$. We define

$$\phi_w = \nu_w - \nu_{w^{-1}}.$$

Proposition 4.2. *The map $\phi_w : G \to \mathbb{R}$ is a quasimorphism with defect $D(\phi_w) \leq 3$.*

Proof. We need the following claim for the proof.

Claim 4.3. *Let $y\mathrm{x}y'$ be a reduced expression where* x *is a vertex element. Then*

$$\nu_w(y\mathrm{x}y') - \nu_w(y) - \nu_w(y') \in \{0, 1\}.$$

Proof. Suppose $\nu_w(y) = n$ with $y = y_0 w_1 y_1 \cdots w_n y_n$ and $\nu_w(y') = n'$ with $y' = y_0' w_1' y_1' \cdots w_{n'}' y_{n'}'$, where $w_i, w_i' \in CwC$. Then

$$y\mathrm{x}y' = y_0 w_1 y_1 \cdots w_n (y_n \mathrm{x} y_0') w_1' y_1' \cdots w_{n'}' y_{n'}'$$

is a reduced decomposition and thus $\nu_w(y\mathrm{x}y') \geq \nu_w(y) + \nu_w(y')$.

On the other hand, suppose that $\nu_w(y\mathrm{x}y') = m$ and we have a reduced decomposition of $y\mathrm{x}y'$ that contains m disjoint copies of words in CwC. We also have a reduced expression of $y\mathrm{x}y'$ induced from arbitrary reduced words representing y and y'. By chopping up the second reduced expression so that subwords have lengths matching the first reduced decomposition, it follows from Corollaries 2.22 and 2.26 that all the subwords in the first expression representing elements in CwC give disjoint subwords of y or y', except when the subword intersects x, which can occur for at most one subword. Thus

$$\nu_w(y\mathrm{x}y') \leq \nu_w(y) + \nu_w(y') + 1,$$

which shows the claim. □

Let $g, h \in G$. Using Propositions 2.23 and 2.27 we see that there are elements $y_1, y_2, y_3 \in G$ and vertex elements $\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3$ such that

$$\begin{aligned} g &= y_1^{-1} \mathrm{x}_1 y_2 \\ h &= y_2^{-1} \mathrm{x}_2 y_3 \\ (gh)^{-1} &= y_3^{-1} \mathrm{x}_3 y_1 \end{aligned}$$

as reduced expressions.

Using Claim 4.3 we see that

$$\begin{aligned} |\phi_w(g) - \nu_w(y_1^{-1}) - \nu_w(y_2) + \nu_{w^{-1}}(y_1^{-1}) + \nu_{w^{-1}}(y_2)| &\leq 1 \\ |\phi_w(h) - \nu_w(y_2^{-1}) - \nu_w(y_3) + \nu_{w^{-1}}(y_2^{-1}) + \nu_{w^{-1}}(y_3)| &\leq 1 \\ |\phi_w(gh) - \nu_w(y_1^{-1}) - \nu_w(y_3) + \nu_{w^{-1}}(y_1^{-1}) + \nu_{w^{-1}}(y_3)| &\leq 1. \end{aligned}$$

Using that $\nu_{w^{-1}}(g) = \nu_w(g^{-1})$ for any g we obtain

$$|\phi_w(g) + \phi_w(h) - \phi_w(gh)| \leq 3,$$

which shows the proposition. □

We can now prove Theorem 4.1 using quasimorphisms:

Proof of Theorem 4.1. Let G be a group which splits over C and let $\sum_{i=1}^n g(1)$ be some integral chain where every term either lies in a vertex group or is cyclically reduced and let $N \in \mathbb{Z}_+$ be some integer. Suppose for some $g = g(i)$ cyclically reduced, the equation $g^N = h^k h' d$ as in the theorem does not hold. Then for $w = g^N$, we know $\nu_{w^{-1}}(g(j)^m) = 0$ for all j and $m \in \mathbb{Z}_+$. It follows that

$$\phi_w(g(j)^m) \geq 0$$

for any $j \in \{1, \dots, n\}$. Moreover,

$$\phi_w(g(i)^m) \geq \left\lfloor \frac{m}{N} \right\rfloor$$

and thus $\bar{\phi}_w(g(i)) \geq \frac{1}{N}$ for the homogenization.

We conclude that

$$\sum_{j=1}^n \bar{\phi}_w(g(j)) \geq \frac{1}{N}.$$

On the other hand, we have that $D(\phi_w) \leq 3$ and thus $D(\bar{\phi}_w) \leq 6$ by Proposition 2.10. By Bavard's Duality Theorem (Theorem 2.11) we obtain

$$\mathrm{scl}_G(\sum_{i=1}^n g(i)) \geq \frac{1}{12N},$$

which completes the proof of Theorem 4.1. □

5. Central/Malnormal Subgroups

In this section we will use Theorem 4.1 to give a criterion for chains in certain amalgamated free products and HNN extensions to have a gap in stable commutator length.

In order to apply Theorem 4.1 we need to solve the following equation for some fixed integer $N \in \mathbb{N}$

$$g^N = h^k h' c, \tag{5.1}$$

where both sides are reduced decompositions (see Definitions 2.20 and 2.24), where $|g| \geq |h|$, h' is a prefix of h, $c \in C$ and $k \geq N$.

In order to solve Equation (5.1), we define and study BCMS-D subgroups H of a group G for an integer D (Definition 5.8). Central subgroups are BCMS-0 and malnormal subgroups are BCMS-1. As a key example, if $\Lambda \subset \Gamma$ is an induced subgraph of a graph Γ then the associated subgroup $\mathrm{A}(\Lambda)$ of the RAAG $\mathrm{A}(\Gamma)$ is a BCMS-D subgroup for some D; see Lemma 6.6.

If the subgroup C that G splits over is BCMS-D, then for $N = D + 2$ we solve Equation (5.1) as follows:

- if $|g| = |h|$ then Equation (5.1) reduces to $g^N = h^N c$ for some $c \in C$. We show that there is some element $z \in C$ which commutes with g such that $g = hz$, so that $c = z^N$; see Proposition 5.19.
- if $|g| > |h|$ then Equation (5.1) implies that there is some element $x \in G$ and an element $c \in C$ which commutes with x such that $g = x^m c$ for some $m \geq 2$; see Proposition 5.21.

In both cases, equation (5.1) only holds when g can be replaced by a simpler equivalent integer chain. This way we show:

Theorem 5.1. *Let G be the fundamental group of a graph of groups such that the embedding of every edge group $C \le G$ has property BCMS-D. Let c be an integral chain in G. Then either c is equivalent (Definition 2.4) to an integral chain $\tilde{c}$ such that every term lies in a vertex group or*

$$\mathrm{scl}_G(c) \ge \frac{1}{12(D+2)}.$$

This section is organized as follows. In Sections 5.1 and 5.2 we define CM-subgroups and BCMS-D subgroups respectively. In Sections 5.3 and 5.4 we prove properties of BCMS-D subgroups related to Equation (5.1). Then we solve Equation (5.1) in Section 5.5 and prove Theorem 5.1 in Section 5.6.

5.1. **CM-subgroups.** In this section we introduce *central/malnormal subgroups (CM-subgroups).* CM-subgroups are generalizations of two very different types of subgroups: central subgroups and malnormal subgroups. Recall that a subgroup $H \le G$ is *central*, if for every element $g \in G$ and every element $h \in H$ we have that $ghg^{-1} = h$. On the other hand, a subgroup $H \le G$ is *malnormal*, if for every element $g \in G \setminus H$ and every element $h \in H$ we have that $ghg^{-1} \notin H$.

We say that an element $g \in G$ is a *CM-representative for $H \le G$*, if for every $h \in H$ either

(i) $ghg^{-1} = h$, or
(ii) $ghg^{-1} \notin H$.

For a subset S of G, let $Z_H(S)$ be the subgroup of elements in H commuting with all elements of S. When $S = \{g\}$, we simply denote it as $Z_H(g)$. Then g is a CM-representative for H if and only if $gHg^{-1} \cap H = Z_H(g)$.

Proposition 5.2 (Uniqueness of CM-representatives)**.** *Let g be a CM-representative for $H \le G$. Then $g' \in HgH$ is a CM-representative if and only if there are elements $h \in H$, $z \in Z_H(Z_H(g))$ such that $g' = hzgh^{-1}$. In this case, we have that $Z_H(g') = hZ_H(g)h^{-1}$.*

Proof. First assume that g is a CM-representative and let $g' = hzgh^{-1}$ for some $h \in H$ and $z \in Z_H(Z_H(g))$. We show that g' is a CM-representative. For any $x \in H$, we have $g'xg'^{-1} = hzg(h^{-1}xh)g^{-1}z^{-1}h^{-1}$. Since $h^{-1}xh \in H$ and g is a CM-representative, either $g(h^{-1}xh)g^{-1} \notin H$ or $g(h^{-1}xh)g^{-1} = h^{-1}xh$. In the former case we have $g'xg'^{-1} \notin H$ since $hz \in H$, while in the latter case we have $h^{-1}xh \in Z_H(g)$ and $g'xg'^{-1} = hz(h^{-1}xh)z^{-1}h^{-1} = h(h^{-1}xh)h^{-1} = x$. Thus g' is a CM-representative, and the calculation shows that $x \in Z_H(g')$ if and only if $h^{-1}xh \in Z_H(g)$, i.e. $x \in hZ_H(g)h^{-1}$.

Conversely, if $g' = h_1gh_2$ is a CM-representative for some $h_1, h_2 \in H$, then by what we proved above, so is $g'' = hg$, where $h = h_2h_1$. Then for any $x \in Z_H(g)$, we have $g''xg''^{-1} = hgxg^{-1}h^{-1} = hxh^{-1} \in H$. Since g'' is a CM-representative, we must have $hxh^{-1} = g''xg''^{-1} = x$ for all $x \in Z_H(g)$. Hence $h \in Z_H(Z_H(g))$. ☐

Definition 5.3 (CM-subgroups and CM-choice)**.** We say that $H \le G$ is a *CM-subgroup of G*, if for every $g \in G$ there is an element $\bar{g} \in HgH$ such that $\bar{g}$ is a CM-representative for H.

A *CM-choice* for a CM-subgroup $H \le G$ is a choice of one CM-representative for each double coset HgH with $g \in G$.

Every central or malnormal subgroup $H \le G$ is a CM-subgroup. The motivating example for CM-subgroups come from right-angled Artin groups: We will see that for any induced subgraph $\Lambda \subset \Gamma$ the associated right-angled Artin group $\mathrm{A}(\Lambda)$ is a CM-subgroup of $\mathrm{A}(\Gamma)$ (Lemma 6.6). We will have this application in mind throughout this section.

Example 5.4. Consider the graph Δ_1 with vertex set $\{v_0, v_1\}$ and empty edge set and the graph Δ_2 with vertex set $\{v_0, v_1, v_2\}$ and a single edge (v_0, v_2). The associated right-angled Artin groups are $\mathrm{A}(\Delta_1) \cong \mathbb{Z} \star \mathbb{Z}$ and $\mathrm{A}(\Delta_2) \cong \mathbb{Z} \star \mathbb{Z}^2$.

The subgroup $\mathrm{A}(\Delta_1)$ arises naturally as a subgroup of $\mathrm{A}(\Delta_2)$ and is neither central nor malnormal, but it is a CM-subgroup by Lemma 6.6. Not every element of $\mathrm{A}(\Delta_2) \setminus \mathrm{A}(\Delta_1)$ is a CM-representative, such as $v_1v_2 \in \mathrm{A}(\Delta_2) \setminus \mathrm{A}(\Delta_1)$: For $v_0 \in \mathrm{A}(\Delta_1)$ we have that $(v_1v_2)v_0(v_1v_2)^{-1} = v_1v_0v_1^{-1} \in \mathrm{A}(\Delta_1)$, but $(v_1v_2)v_0(v_1v_2)^{-1} \ne v_0$. However, $v_2 \in \mathrm{A}(\Delta_1)(v_1v_2)\mathrm{A}(\Delta_1)$ *is* a CM-representative.

We will see that for every double coset $\mathrm{A}(\Delta_1)g\mathrm{A}(\Delta_1)$, an element with the shortest word length in the double coset is a CM-representative (Lemma 6.6). This yields a natural CM-choice.

Proposition 5.5 (Inheritance properties of CM-subgroups)**.** *Let $K \le H \le G$ be nested subgroups.*

- *If $K \leq G$ is a CM-subgroup then $K \leq H$ is a CM-subgroup.*
- *If $K \leq H$ is a CM-subgroup and $H \leq G$ is a CM-subgroup then $K \leq G$ is a CM-subgroup.*

Proof. The first item is immediate. For the second item, for any $g \in G \backslash K$ we need to find a CM-representative in KgK. As $H \leq G$ is a CM-subgroup, there is a CM-representative in HgH for $H \leq G$. By Proposition 5.2 there is a CM-representative of the form $\bar{g} = gh$ for some $h \in H$. Similarly, since $K \leq H$ is a CM-subgroup, we have a CM-representative $\bar{h} = kh$ for h with $k \in K$.

Then $g' = gk^{-1} = \bar{g}\bar{h}^{-1}$ is a CM-representative in KgK. Indeed, for any $k_0 \in K$, we have

$$g' k_0 g'^{-1} = \bar{g}\bar{h}^{-1} k_0 \bar{h} \bar{g}^{-1}.$$

As $g' k_0 g'^{-1}$ is the conjugate of $\bar{h}^{-1} k_0 \bar{h} \in H$ by $\bar{g}$, it is either outside H and hence outside K or equal to $\bar{h}^{-1} k_0 \bar{h}$. In the latter case, either $\bar{h}^{-1} k_0 \bar{h} \notin K$ or $\bar{h}^{-1} k_0 \bar{h} = k_0$ since $\bar{h}$ is a CM-representative. □

5.2. **BCMS-D subgroups.** Given a proper CM-subgroup H of a group G and a CM-representative $g \in G \setminus H$ the centralizer $Z_H(g)$ measures how much the subgroup $H < G$ fails to be malnormal for the element g. It has an interesting structure in the motivating example of RAAGs.

Example 5.6. Let Δ_1 and Δ_2 be the graphs defined in Example 5.4. We have seen that $v_2 \in \mathrm{A}(\Delta_2) \setminus \mathrm{A}(\Delta_1)$ is a CM-representative. Here $Z_{\mathrm{A}(\Delta_1)}(v_2) = \mathrm{A}(\Delta_0)$ where Δ_0 is the graph with single vertex v_0.

More generally, we will see that if we choose CM-representatives to be elements in each double coset of minimal length then every such centralizer is the right-angled Artin group on an induced subgraph of the defining graph (Lemma 6.6) and thus it is again a CM-subgroup.

On the other hand, if we choose the CM-representatives in a different way, the centralizers may not have this structure, but they only differ by conjugations according to Proposition 5.2

Let H_0 be a group and let H_1 be a proper CM-subgroup of H_0. Let $h_0 \in H_0 \setminus H_1$ be a CM-representative. Then $H_2 := Z_{H_1}(h_0)$ is a subgroup of H_1. There are three cases:

(i) if $H_2 = H_1$ then H_1 lies in the centralizer of the element h_0,
(ii) if $H_2 = \{e\}$, then H_1 behaves like a malnormal subgroup with respect to the element h_0, or
(iii) $\{e\} \neq H_2 < H_1$ is a proper nontrivial subgroup.

If h_0 is as in case (iii) and H_2 is a CM-subgroup of H_1 then we may continue this process: Given a CM-representative $h_1 \in H_1 \setminus H_2$, define $H_3 = Z_{H_2}(h_1)$.

Informally, if this process always yields CM-subgroups and eventually stops (in about D steps), then we say the subgroup H_1 has *bounded CM-subgroup sequence of depth* D, which we abbreviate as BCMS-D. We make this precise in the following definitions.

Definition 5.7 (CM-subgroup sequence)**.** In a group H, a CM-subgroup sequence of length $m+1$ is a sequence of nested subgroups $H = H_0 > H_1 > \cdots > H_{m+1} \geq H_{m+2}$ such that H_{i+1} is a proper CM-subgroup of H_i for all $0 \leq i \leq m$ (*not* including $i = m+1$) and $H_{i+2} = Z_{H_{i+1}}(g_i)$ for some $g_i \in H_i \setminus H_{i+1}$.

For any CM-subgroup sequence $H_{m+2} \leq \cdots \leq H_0$, if H_1 is central we must have $H_2 = H_1$, which forces $m = 0$. If H_1 is malnormal, then we have $H_2 = \{e\} = H_3$, forcing $m \leq 1$. Note that not every nested sequence of proper CM-subgroups is a CM-subgroup sequence due to the requirement $H_{i+2} = Z_{H_{i+1}}(g_i)$. For instance, $\{e\} \leq \mathbb{Z} \leq \mathbb{Z}^2$ is a nested sequence of proper CM-subgroups, but $Z_{\mathbb{Z}}(g) \neq \{e\}$ for all $g \in \mathbb{Z}^2$.

It is important to note that, in the definition of CM-subgroup sequences, the only requirement on H_{m+2} is that $H_{m+2} = Z_{H_{m+1}}(g_m)$ for some $g_m \in H_m \setminus H_{m+1}$, and in general it may not be a CM-subgroup of H_{m+1}. It is part of the definition of BCMS subgroups below that H_{m+2} is required to be a proper CM-subgroup of H_{m+1} except when $H_{m+2} = H_{m+1}$.

Definition 5.8 (BCMS-D)**.** Let $D \in \mathbb{Z}_{\geq 0}$ be an integer, and let H_0 be a group. We say that a subgroup $H_1 \leq H_0$ (or really the pair (H_0, H_1)) has *bounded CM-subgroup sequences of depth D (BCMS-D)* if H_1 is a CM-subgroup and for every CM-subgroup sequence $H_{m+2} \leq \cdots \leq H_0$ we have that either $H_{m+2} = H_{m+1}$ or that $H_{m+2} < H_{m+1}$ is a proper CM-subgroup. Moreover we require that every CM-subgroup sequence has length at most $D+1$, i.e. if $H_{m+2} \leq \cdots \leq H_0$ is a CM-subgroup sequence then $m \leq D$. We also say H_1 is a BCMS-D subgroup.

We see that central subgroups have BCMS-0 and malnormal subgroups have BCMS-1. Note that by definition a subgroup H has BCMS-D then it also has BCMS-D' if $D' \geq D$, i.e. we do not require D to be the optimal upper bound.

In general, verifying whether a CM-subgroup $H \le G$ has BCMS-D requires one to check all CM-subgroup sequences. As we saw in Example 5.6, certain choices of CM-representatives have centralizers that are easier to study in some cases. We will show that one can restrict attention to some special families of CM-subgroup sequences corresponding to nice choices of CM-representatives to verify whether a CM-subgroup has BCMS-D.

We incorporate the choice of CM-representatives into the following notion.

Definition 5.9 (CM-subgroup-choice)**.** A *CM-subgroup-choice* $\mathcal{I}(G)$ is a CM-choice (Definition 5.3) for every proper CM-subgroup $H < G$.

Given a CM-subgroup-choice $\mathcal{I}(G)$, whenever we have a chain of proper subgroups $K < H < G$ such that $K < H$ and $H < G$ are CM-subgroups. Then $K < G$ is also a CM-subgroup by Proposition 5.5. For any element $h \in H \setminus K$ the CM-subgroup-choice $\mathcal{I}(G)$ gives us a CM-representative of h for $K < G$ which is also a CM-representative for $K < H$.

Definition 5.10 (CM-sequence)**.** Given a proper CM-subgroup $H_1 < H_0$ and a CM-subgroup-choice $\mathcal{I}(H_0)$, a *CM-sequence* of length $m+1$ is a sequence of elements $(h_0, \dots, h_m)$ in G such that there is a CM-subgroup sequence $H_{m+2} \le H_{m+1} < \cdots < H_0$ satisfying

- $h_i \in H_i \setminus H_{i+1}$ is the CM-representative for H_{i+1} provided by $\mathcal{I}(G)$ for all $0 \le i \le m$, and
- $H_{i+2} = Z_{H_{i+1}}(h_i)$ for all $0 \le i \le m$.

Given a *CM-sequence* $(h_0, \dots, h_m)$, it uniquely determines the CM-subgroup sequence $H_{m+2} \le H_{m+1} < \cdots < H_0$ by the relation $H_{i+2} = Z_{H_{i+1}}(h_i)$, which we refer to as the associated CM-subgroup sequence.

Apparently, if H_1 is a BCMS-D subgroup, then any CM-sequence $(h_0, \dots, h_m)$ has length at most $D+1$, i.e. $m \le D$. Conversely, given a CM-subgroup-choice $\mathcal{I}(G)$, not every CM-subgroup sequence appears as one associated to some CM-sequence $(h_0, \dots, h_m)$. However, it suffices to consider CM-subgroup sequences associated to CM-sequences to show that H_1 is a BCMS-D subgroup.

Proposition 5.11. *Fix a CM-subgroup-choice $\mathcal{I}(H_0)$. Suppose H_1 is a proper CM-subgroup of H_0, and for the CM-subgroup sequence $H_{m+2} \le \cdots \le H_0$ associated to any CM-sequence $(h_0, \dots, h_m)$, we have that either $H_{m+2} = H_{m+1}$ or that $H_{m+2} < H_{m+1}$ is a proper CM-subgroup. Then H_1 has BCMS-D if and only if every CM-sequence $(h_0, \dots, h_m)$ has $m \le D$.*

Proof. Given any CM-subgroup sequence $H_{m+2} \le \cdots \le H_0$, we claim that for any $0 \le k \le m$, there is a CM-subgroup sequence $H'_{m+2} \le \cdots \le H'_0$ with $H'_1 = H_1$ and $H'_0 = H_0$ such that

- $H_{m+2} = H_{m+1}$ if and only if $H'_{m+2} = H'_{m+1}$, and $H_{m+2} \le H_{m+1}$ is a proper CM-subgroup if and only if $H'_{m+2} \le H'_{m+1}$ is a proper CM-subgroup;
- there is a CM-sequence $(\bar{h}_0, \cdots, \bar{h}_k)$ whose associated CM-subgroup is $H'_{k+2} \le \cdots \le H'_0$.

The claim with $k = m$ together with our assumption shows that, whenever we have a CM-subgroup sequence $H_{m+2} \le \cdots \le H_0$, we have that either $H_{m+2} = H_{m+1}$ or that $H_{m+2} \le H_{m+1}$ is a proper CM-subgroup. Moreover, there is a CM-sequence $(\bar{h}_0, \cdots, \bar{h}_m)$ of the same length, which proves the proposition.

Thus it suffices to prove this claim, which we show by induction on k. For the base case $k = 0$, by definition there is a CM-representative h_0 for $H_1 < H_0$ (not necessarily from $\mathcal{I}(H_0)$) such that $Z_{H_1}(h_0) = H_2$. Let $\bar{h}_0$ be the CM-representative in $H_1 h_0 H_1$ chosen by $\mathcal{I}(H_0)$. By Proposition 5.2, there is some $h \in H_1$ and $z \in Z_{H_1}(H_2)$ such that $\bar{h}_0 = hzh_0h^{-1}$. In this case, $hH_{m+2}h^{-1} \le \cdots \le hH_2h^{-1} \le H_1 \le H_0$ is a CM-subgroup sequence where $hH_2h^{-1} \le H_1 \le H_0$ is the CM-subgroup sequence associated to the CM-sequence $(\bar{h}_0)$ since $Z_{H_1}(\bar{h}_0) = hH_2h^{-1}$ by Proposition 5.2.

Suppose the claim holds for some $0 \le k < m$, i.e. there is a CM-subgroup sequence $H'_{m+2} \le \cdots \le H'_0$ with $H'_1 = H_1$ and $H'_0 = H_0$, such that the relation between H'_{m+2} and H'_{m+1} corresponds to the relation between H_{m+2} and H_{m+1}, and there is a CM-sequence $(\bar{h}_0, \cdots, \bar{h}_k)$ whose associated CM-subgroup is $H'_{k+2} \le \cdots \le H'_0$. Since $k < m$, there is a CM-representative $h_{k+1} \in H'_{k+1} \setminus H'_{k+2}$ such that $Z_{H'_{k+2}}(h_{k+1}) = H'_{k+3}$. Let $\bar{h}_{k+1} = hzh_{k+1}h^{-1}$ be the CM-representative in $H'_{k+2}h_{k+1}H'_{k+2}$, where $h \in H'_{k+2}$ and $z \in Z_{H'_{k+2}}(H'_{k+3})$. Then $hH'_{m+2}h^{-1} \le \cdots \le hH'_{k+3}h^{-1} \le H'_{k+2} \le \cdots \le H'_0$ is a CM-subgroup sequence where $hH'_{k+3}h^{-1} \le H'_{k+2} \le \cdots \le H'_0$ is the CM-subgroup sequence associated to the CM-sequence $(\bar{h}_0, \cdots, \bar{h}_k, \bar{h}_{k+1})$ since $Z_{H'_{k+2}}(\bar{h}_{k+1}) = hH'_{k+3}h^{-1}$ by Proposition 5.2. This completes the induction and proves the proposition. $\square$

In what follows, we will use the proposition above as an alternative definition of BCMS-D subgroups since it is easier to check. In practice, only certain subgroups arise as H_i in some CM-subgroup sequence associated to a CM-subgroup, and thus one only needs to fix the CM-subgroup-choice for these CM-subgroups of H_0. See the example below.

For every $D \in \mathbb{Z}_+$ there is a BCMS-D subgroup of a group which is not a BCMS-$(D-1)$-subgroup.

Example 5.12. For $n \in \mathbb{N}$, let Δ_n be the graph with vertex and edge set

$$\begin{aligned} V(\Delta_n) &= \{v_0, \dots, v_n\} \text{ and} \\ E(\Delta_n) &= \{(v_i, v_j) \mid |i-j| \geq 2\}. \end{aligned}$$

For $n \in \mathbb{N}$ and $i \in \{1, \dots, n\}$ let Δ_n^i be the induced subgraph of Δ_n with vertex set

$$V(\Delta_n^i) = \{v_i, \dots, v_n\}.$$

By Lemma 6.6 we have that $\mathrm{A}(\Delta_n^1) < \mathrm{A}(\Delta_n)$ is a CM-subgroup and that v_0 is a CM-representative. We compute that $Z_{\mathrm{A}(\Delta_n^1)}(v_0) = \mathrm{A}(\Delta_n^2)$. More generally we will see that $\mathrm{A}(\Delta_n^i)$ is a CM-subgroup of $\mathrm{A}(\Delta_n^{i-1})$, that v_{i-1} is a CM-representative and that $Z_{\mathrm{A}(\Delta_n^i)}(v_{i-1}) = \mathrm{A}(\Delta_n^{i+1})$ for $1 \leq i \leq n-1$.

Thus $(v_0, \dots, v_n)$ is a CM-sequence of length $n+1$ and the associated CM-subgroup sequence is

$$\{e\} \leq \{e\} \leq \mathrm{A}(\Delta_n^n) \leq \dots \leq \mathrm{A}(\Delta_n^1) \leq \mathrm{A}(\Delta_n).$$

We will see that those are, in some sense, the longest CM-sequence for subgroups associated to induced subgraphs on RAAGs (Lemma 6.8).

5.3. **Normal forms for elements in BCMS-D subgroups.** If H_1 is a BCMS-D subgroup of H_0, given a CM-subgroup-choice $\mathcal{I}(H_0)$, then we may write every element as a product of CM-representatives up to conjugation as follows:

Proposition 5.13 (Normal form for elements). *Let H_1 be a BCMS-D subgroup of H_0 with a CM-subgroup-choice $\mathcal{I}(H_0)$, and let $g \in H_0 \setminus H_1$ be an element.*

Then there is $n \leq D$, a CM-sequence $(h_0, \dots, h_n)$ with associated CM-subgroup sequence $H_{n+2} \leq \dots \leq H_0$, and a conjugate g' of g by an element of H_1 such that

$$g' = h_0 \cdots h_n e_n$$

with $e_n \in H_{n+2}$. Moreover, the integer n and the CM-sequence $(h_0, \cdots, h_n)$ are uniquely determined by g, and e_n is unique up to conjugation in H_{n+2}.

Proof. We inductively prove the following statement:

Claim 5.14. *For every $m \geq 0$, there is a conjugate g' of g by an element of H_1 such that either*

(i) $g' = h_0 \cdots h_m e_m$ for some $e_m \in H_{m+1}$, where $(h_0, \dots, h_m)$ is a CM-sequence with $H_{m+2} \leq \dots \leq H_0$ as the associated CM-subgroup sequence, or

(ii) $g' = h_0 \cdots h_j e_j$ for some $e_j \in H_{j+2}$ and $j \leq m$, where $(h_0, \dots, h_j)$ is a CM-sequence with $H_{j+2} \leq \dots \leq H_0$ as the associated CM-subgroup sequence.

Proof. Let $h_0 = hgh'$ be the CM-representative in $H_1 g H_1$ provided by $\mathcal{I}(H_0)$, where $h, h' \in H_1$. Then $h_0 = g'hh'$ for $g' := hgh^{-1}$, and thus $g' = h_0 e_0$ with $e_0 := (hh')^{-1} \in H_1$. This shows the claim for $m = 0$.

Suppose the claim is true for some $m \geq 0$. If statement (ii) holds for m then it also holds for $m+1$ and we are done. Thus assume that $g' = h_0 \cdots h_m e_m$ where g' is a conjugate of g by some element in H_1, $(h_0, \dots, h_m)$ is a CM-sequence with associated CM-subgroup sequence $H_{m+2} \leq \dots \leq H_0$, and $e_m \in H_{m+1}$.

If $e_m \in H_{m+2}$ we are done as in case (ii) as well. Otherwise, let $h_{m+1} \in H_{m+1} \setminus H_{m+2}$ be the CM-representative in $H_{m+2} e_m H_{m+2}$ given by $\mathcal{I}(H_0)$. Then $h^l h_{m+1} h^r = e_m$ for some $h^l, h^r \in H_{m+2}$. Thus

$$g' = h_0 \cdots h_m h^l h_{m+1} h^r = h^l h_0 \cdots h_m h_{m+1} h^r$$

as h^l commutes with all $h_0, \dots, h_m$ by the definition of H_{m+2}. Conjugating both sides of the equation above by h^l and setting $e_{m+1} = h^r h^l$ proves the claim. □

Now as $H_1 < H_0$ has property BCMS-D we will arrive at item (ii) of the claim eventually (if $m \geq D$).

The uniqueness can be observed in the inductive construction above as follows. Note that h_0 is uniquely determined as the CM-representative in $H_1 g H_1$ since $\mathcal{I}(H_0)$ is fixed. Next we show that e_0 is uniquely determined up to conjugation by an element in H_2. Suppose there is a different choice $e_0' \in H_1$ such that $h_0 e_0' = hh_0 e_0 h^{-1}$ for some $h \in H_1$, then $h^{-1} = h_0[e_0 h^{-1}(e_0')^{-1}]h_0^{-1}$, which forces $h^{-1} = e_0 h^{-1}(e_0')^{-1}$ as h_0 is a CM-representative for H_1. Thus $h^{-1}e_0' h = e_0$, so $h_0 e_0' = hh_0 e_0 h^{-1} = hh_0 h^{-1} e_0'$, which implies $h_0 = hh_0 h^{-1}$, i.e. $h \in Z_{H_1}(h_0) = H_2$. This proves that e_0' differs from e_0 via conjugation by some $h \in H_2$. In particular, the double coset $H_2 e_0 H_2$ is uniquely determined and so is h_1. Continuing this process, one can observe that each h_i in the expression is uniquely determined, each element e_i is unique up to conjugation by an element of H_{i+2}, and the integer n is characterized as the first n such that $e_n \in H_{n+2}$ (which is not ambiguous by the uniqueness up to conjugation). □

Definition 5.15 (CM-reduced element)**.** Suppose that H_1 is a BCMS-D subgroup of H_0 with CM-subgroup-choice $\mathfrak{I}(H_0)$. For any $g \in H_0 \setminus H_1$, we say that g is *CM-reduced* if we have $e_n = 1$ when g is written as in the normal form given by Proposition 5.13 and no conjugation is involved. That is, $g = h_0 \cdots h_n$ for a CM-sequence $(h_0, \cdots, h_n)$ and some $0 \le n \le D$.

Proposition 5.16. *Let H_1 be a BCMS-D subgroup of H_0 with CM-subgroup-choice $\mathfrak{I}(H_0)$. Let $g \in H_0$ be an element and let g', e_n and $(h_0, \dots, h_n)$ be as in the normal form from Proposition 5.13. Then g is conjugate to he_n by an element of H_1 and equivalent to $h + e_n$ as a chain, where $h = h_0 \cdots h_n$ is CM-reduced.*

Proof. This follows immediately from Proposition 5.13 and Definition 2.4 noting that e_n commutes with $h_0, \dots, h_n$. □

5.4. **Equations in amalgamated free products or HNN extensions.** In the rest of Section 5, we will consider a group G that splits over a BCMS-D subgroup C. Note that if $g \in G \setminus C$ can be written as a cyclically reduced word in the sense of Definitions 2.20 or 2.24, then naturally any element $cgc' \in CgC$ also has this property. In particular, in this case, any CM-representative in CgC with respect to the CM-subgroup C can be written as a cyclically reduced word.

Similarly, if g is CM-reduced with $g = c_0 \cdots, c_m$ for a CM-sequence $(c_0, \cdots, c_m)$, then g is cyclically reduced if and only if c_0 is. So we say g is cyclically reduced and CM-reduced (e.g. in Proposition 5.19 below) if g can be written this way with c_0 cyclically reduced.

We will need the following proposition to compare terms in certain expressions in a group G that splits over a BCMS-D subgroup C. This is similar to Corollaries 2.22 and 2.26.

Proposition 5.17. *Let G be a group that splits over a BCMS-D subgroup C. Let $\mathfrak{I}(G)$ be a CM-subgroup-choice. For some $m \le D$ let $(c_0, \dots, c_m)$ is a CM-sequence and let $C_{m+2} \le \cdots \le C_1 := C \le C_0 := G$ be the associated CM-subgroup-sequence.*

Suppose $c_0 \in G \setminus C$ (or equivalently g) can be written as a cyclically reduced word. Let $n \ge m+2$ and suppose there are elements $x_1, \dots, x_{n-1}, x_1', \dots, x_{n-1}' \in C_{m+1}$ and $x_0, x_0', x_n, x_n' \in C_1$ such that

$$x_0 c^{(m)} x_1 \cdots c^{(m)} x_n = x_0' c^{(m)} x_1' \cdots c^{(m)} x_n',$$

where $c^{(m)} = c_0 \cdots c_m$. Then there are $d_1, \dots, d_{n-m} \in C_{m+2}$ such that

$$d_{i-1} x_i' d_i^{-1} = x_i$$

for all $2 \le i \le n-m$.

Proof. We observe that by Corollaries 2.22 and 2.26 there are elements $d_0, \dots, d_n \in C_1$ with $d_0 = e = d_n$ such that $x_0 c^{(m)} x_1 = d_0 x_0' c^{(m)} x_1' d_1^{-1}$ and $c^{(m)} x_i = d_{i-1} c^{(m)} x_i' d_i^{-1}$ for all $i \in \{2, \dots, n\}$.

Claim 5.18. *For every $j \in \{0, \dots, m\}$ we have that $d_i \in C_{j+2}$ for all $i \in \{1, \dots, n-j\}$.*

Proof. We proceed by induction. For $j = 0$ we write $c^{(m)} = c_0 c''$ with $c'' = c_1 \cdots c_m$. Then we obtain

$$c_0^{-1} d_{i-1} c_0 = c'' x_i d_i {x_i'}^{-1} {c''}^{-1}$$

for all $i \in \{2, \dots, n\}$ from $c^{(m)} x_i = d_{i-1} c^{(m)} x_i' d_i^{-1}$. Observe that both $d_{i-1} \in C_1$ and $c'' x_i d_i {x_i'}^{-1} {c''}^{-1} \in C_1$. Since c_0 is a CM-representative for $C_1 < C_0$ we have $d_{i-1} \in C_2 = Z_{C_1}(c_0)$. Since $d_n = e$ we conclude that $d_i \in C_2$ for all $i \in \{1, \dots, n\}$.

Suppose the claim is true for some $j-1 \in \{0, \dots, m-1\}$. We wish to show that it is true for j as well. We may write $c^{(m)} = c' c_j c''$ for $c' = c_0 \cdots c_{j-1}$ and $c'' = c_{j+1} \cdots c_m$. By the induction hypothesis we have that $d_i \in C_{j+1}$ for all $i \in \{1, \dots, n-j+1\}$, so all such d_i commute with c'. As above we obtain

$$c_j^{-1} d_{i-1} c_j = c'' x_i d_i {x_i'}^{-1} {c''}^{-1}$$

for all $i \in \{2, \ldots, n-j+1\}$ from $c^{(m)}x_i = d_{i-1}c^{(m)}x_i'd_i^{-1}$. Note that $d_{i-1}, d_i, x_i, x_i', c'' \in C_{j+1}$ for all $i \in \{2, \ldots, n-j+1\}$. Thus, as c_j is a CM-representative, we see that $d_i \in C_{j+2}$ for all $i \in \{1, \ldots, n-j\}$. This completes the induction. □

For $j = m$ the claim implies that $d_i \in C_{m+2}$ for all $i \in \{1, \ldots, n-m\}$ and thus all such d_i commute with $c^{(m)}$. Hence from $c^{(m)}x_i = d_{i-1}c^{(m)}x_i'd_i^{-1}$ we have

$$x_i = d_{i-1}x_i'd_i^{-1}$$

for all $i \in \{2, \ldots, n-m\}$. This finishes the proof. □

5.5. **Solutions to Equation (5.1).**

Proposition 5.19. *Let G be a group that splits over a BCMS-D subgroup C, and let $\mathcal{I}(G)$ be a CM-subgroup choice. Suppose $g \in G$ is cyclically reduced and CM-reduced. Suppose there is a cyclically reduced word $h \in G$ with $g^N = h^N c$ for some $c \in C$ and $N \geq D+2$. Then there is an element z which commutes with g such that $g = hz$.*

Proof. By our assumption, we have that $g = c_0 \cdots c_m$ where c_0 is cyclically reduced and $(c_0, \ldots, c_m)$ is a CM-sequence with associated CM-subgroup sequence $C_{m+2} \leq \cdots \leq C_0$, where $C_0 = G$ and $C_1 = C$. Note that $m \leq D$ since C is a BCMS-D subgroup.

By Corollaries 2.22 and 2.26 there are $d_i \in C$ for $0 \leq i \leq N$ with $d_0 = e = d_N$ such that $g = d_{i-1}hd_i^{-1}$ for $1 \leq i \leq N-1$ and $g = d_{N-1}hcd_N^{-1}$. Redefining d_N^{-1} to be cd_N^{-1} we get that $g = d_{i-1}hd_i^{-1}$ for $1 \leq i \leq N$ and thus

$$d_{i-1}^{-1}gd_i = d_i^{-1}gd_{i+1} \tag{5.2}$$

for all $1 \leq i \leq N-1$.

Claim 5.20. *For every $0 \leq j \leq m+1$ we have that $d_i \in C_{j+1}$ for all $0 \leq i \leq N-j$.*

Proof. We proceed by induction. For $j = 0$ the claim is immediate as all terms are in $C_1 = C$.

Suppose the claim is true for some $0 \leq j \leq m$. Write $g = c_0 \cdots c_m = c'c_jc''$ for $c' = c_0 \cdots c_{j-1}$ and $c'' = c_{j+1} \cdots c_m$. Observe that by the induction hypothesis, c' commutes with d_i for all $0 \leq i \leq N-j$. Thus for all $1 \leq i \leq N-j-1$, we deduce from equation (5.2) that

$$c_j^{-1}\left(d_id_{i-1}^{-1}\right)c_j = c''d_{i+1}d_i^{-1}c''^{-1}.$$

By the induction hypothesis, $c''d_{i+1}d_i^{-1}c''^{-1} \in C_{j+1}$ for all such i. Thus $d_id_{i-1}^{-1} \in C_{j+2}$ for all $1 \leq i \leq N-j-1$ since c_j is a CM-representative. Recall that $d_0 = e$. Thus for every $i \in \{1, \ldots, N-j-1\}$ we have that

$$d_i = d_id_0^{-1} = (d_id_{i-1}^{-1})(d_{i-1}d_{i-2}^{-1}) \cdots (d_1d_0^{-1}) \in C_{j+2}.$$

This shows the claim. □

In particular for $j = m+1$ the claim implies that $d_i \in C_{m+2}$ for all $0 \leq i \leq N-m-1$. Since $m \leq D$ and $D+2 \leq N$, we have that $d_1 \in C_{m+2}$. Thus d_1 commutes with g. This concludes the proof of Proposition 5.19 as $g = d_0^{-1}hd_1 = hd_1$. □

Proposition 5.21. *Let G be a group that splits over a BCMS-D subgroup C, and let $\mathcal{I}(G)$ be a CM-subgroup-choice. Let $g, h \in G$ be cyclically reduced words with $|g| > |h|$ and let h' be a prefix of h. Suppose*

$$g^N = h^kh'c$$

for some $c \in C$ and $N \geq D+2$.

Then there is a cyclically reduced element $x \in G$ such that $g = x^{n_g}c$ for some $n_g \geq 2$ and $c \in C$ that commutes with x.

Proof. Let $C_0 = G$ and $C_1 = C$. We inductively prove the following claim:

Claim 5.22. *There are two coprime integers $n_g, n_h \in \mathbb{Z}_+$ and $0 \leq n_h' < n_h$ such that for every $m \geq 0$ either* *(i) there is a CM-sequence* $(c_0, \cdots, c_m)$ *with the associated CM-subgroup sequence* $C_{m+2} \le \cdots \le C_0$ *and elements* $d_g, d_h \in C$ *such that*

$$\begin{aligned} d_g g d_g^{-1} &= c^{(m)} z_1 \cdots c^{(m)} z_{n_g}, \text{ and} \\ d_h h d_h^{-1} &= c^{(m)} z'_1 \cdots c^{(m)} z'_{n_h}, \end{aligned}$$

for $c^{(m)} = c_0 \cdots c_m$ *and* $z_i, z'_i \in C_{m+1}$*, or*

(ii) there is an $n \le m$ *and a CM-sequence* $(c_0, \dots, c_n)$ *with the associated CM-subgroup-sequence* $C_{n+2} \le \cdots \le C_0$ *and elements* $d_g, d_h \in C$ *such that*

$$\begin{aligned} d_g g d_g^{-1} &= c^{(n)} z_1 \cdots c^{(n)} z_{n_g}, \text{ and} \\ d_h h d_h^{-1} &= c^{(n)} z'_1 \cdots c^{(n)} z'_{n_h}, \end{aligned}$$

for $c^{(n)} = c_0 \cdots c_n$ *and* $z_i, z'_i \in C_{n+2}$.

Proof. We first show that the claim is true for $m = 0$. Let d be the greatest common divisor of $|g|$ and $|h|$. Note that c lies in C, which is the subgroup that G splits over, so it can be ignored whenever we measure the length of a reduced word. Since both g and h are cyclically reduced and h' is a prefix of h, we have $N|g| = |g^N| = |h^k h'| = k|h| + |h'|$. Hence d also divides $|h'|$. Thus we can write $g = g_1 \cdots g_{n_g}$, $h = h_1 \cdots h_{n_h}$ and $h' = h_1 \cdots h_{n'_h}$, where $n_g = |g|/d$, $n_h = |h|/d$, $n'_h = |h'|/d$ and all the g_i and h_i are reduced words of length d. Note that $n_g > n_h \ge 1$ since $|h| < |g|$.

Then we have reduced decompositions

$$(g_1 \cdots g_{n_g})^N = (h_1 \cdots h_{n_h})^k h_1 \cdots h_{n'_h - 1}(h_{n'_h} c).$$

By Corollaries 2.22 and 2.26 there are elements $d_0, \dots, d_{N n_g} \in C_1 = C$ with $d_0 = e$ and $d_{N n_g} = c^{-1}$ such that $g_i = d_{i-1} h_i d_i^{-1}$ for all $1 \le i \le N n_g$, where the index i in g_i and h_i is taken mod n_g and n_h respectively. Thus for all $1 \le i \le n_g$, we have

$$g_i = d_{i-1} h_i d_i^{-1} = d_{i-1} h_{i+n_h} d_i^{-1} = d_{i-1} d_{i+n_h-1}^{-1} g_{i+n_h} d_{i+n_h} d_i^{-1},$$

and hence $g_i \in C g_{i+n_h} C$. As n_h and n_g are coprime we see that $g_i \in C g_1 C$ for all $1 \le i \le n_g$, and by $g_i = d_{i-1} h_i d_i^{-1}$ we have $h_i \in C g_1 C$ for all $1 \le i \le n_h$. Let $c_0 \in C$ be the CM-representative of $C g_1 C$ provided by $\mathcal{I}(G)$. Then the above calculations show that

$$\begin{aligned} g &= z_0 c_0 z_1 \cdots c_0 z_{n_g}, \text{ and} \\ h &= z'_0 c_0 z'_1 \cdots c_0 z'_{n_h}, \end{aligned}$$

for some $z_i, z'_i \in C = C_1$. Conjugating g and h by z_0 and z'_0 respectively and possibly changing z_{n_g} and z'_{n_h} we achieve case (i) of the claim with $m = 0$. Note that c_0 is cyclically reduced by the expression above since g is cyclically reduced and $|g| = n_g |g_1| = n_g |c_0|$.

Now suppose that the claim is true for some $m \ge 0$. We prove it for $m + 1$. If item (ii) of the claim holds for m then clearly it holds for $m+1$ and we are done. Thus suppose that item (i) holds for m. We will argue similarly as in the case of $m = 0$. By the induction hypothesis we have that

$$\begin{aligned} g &= d_g^{-1} c^{(m)} z_1 \cdots c^{(m)} z_{n_g} d_g, \text{ and} \\ h &= d_h^{-1} c^{(m)} z'_1 \cdots c^{(m)} z_{n_h} d_h, \end{aligned}$$

for some $d_g, d_h \in C_1$, $c^{(m)} = c_0 \cdots c_m$, and $z_i, z'_i \in C_{m+1}$. Since h' is a prefix of h, we have a reduced decomposition $h = h' \cdot h''$ for some reduced word h''. Comparing it to the reduced decomposition

$$h = \left(d_h^{-1} c^{(m)} z'_1 \cdots c^{(m)} z_{n'_h} \right) \left(c^{(m)} z_{n'_h+1} c^{(m)} z_{n_h} d_h \right),$$

by Corollaries 2.22 and 2.26 we observe that $h' = d_h^{-1} c^{(m)} z'_1 \cdots c^{(m)} z_{n'_h} d_{h'}$ for some $d_{h'} \in C_1$. Thus

$$d_g^{-1} \left(c^{(m)} z_1 \cdots c^{(m)} z_{n_g} \right)^N d_g = d_h^{-1} \left(c^{(m)} z'_1 \cdots c^{(m)} z'_{n_h} \right)^k \left(c^{(m)} z'_1 \cdots c^{(m)} z'_{n'_h} \right) d_{h'} c.$$

Applying Proposition 5.17 to this equation with $n = N \cdot n_g$, we obtain elements $d_1, \cdots, d_{N n_g - m} \in C_{m+2} = Z_{C_{m+1}}(c_m)$ such that $z_i = d_{i-1} z'_i d_i^{-1}$ for all $i \in \{2, \dots, N n_g - m\}$, where the index i in z_i and z'_i is taken mod n_g and n_h respectively.

Note that $m \leq D$ since $(c_0, \ldots, c_m)$ is a CM-sequence, and thus $m+2 \leq D+2 \leq N$. It follows that $(N-2)n_g \geq m \cdot n_g > m$ since $n_g \geq 2$. That is, we have $2n_g < Nn_g - m$ and thus $n_g + 1 + n_h \leq Nn_g - m$ as $|h| < |g|$.

Hence

$$z_i = d_{i-1} z_i' d_i^{-1} = d_{i-1} z_{i+n_h}' d_i^{-1} = d_{i-1} d_{i+n_h-1}^{-1} z_{i+n_h} d_{i+n_h} d_i^{-1}$$

for all $2 \leq i \leq n_g + 1$ where indices in z_i are taken mod n_g. As n_h and n_g are coprime we see that $z_i \in C_{m+2} z_1 C_{m+2}$ for all $1 \leq i \leq n_g$. Combining with $z_i = d_{i-1} z_i' d_i^{-1}$ we have $z_i' \in C_{m+2} z_1 C_{m+2}$ for all $1 \leq i \leq n_h$.

If $z_1 \in C_{m+2}$ then all $z_i, z_i' \in C_{m+2}$ and we achieve item (ii) of the claim with $n = m$ and thus we are done.

Otherwise, let $c_{m+1} \in C_{m+1} \setminus C_{m+2}$ be the CM-representative of $C_{m+2} z_1 C_{m+2}$ provided by $\mathcal{I}(G)$. Using the fact that elements in C_{m+2} commute with $c^{(m)}$, it follows that there are $y_i, y_i' \in C_{m+2}$ such that

$$\begin{aligned} d_g g d_g^{-1} &= y_0 c^{(m+1)} y_1 \cdots c^{(m+1)} y_{n_g}, \text{ and} \\ d_h h d_h^{-1} &= y_0' c^{(m+1)} y_1' \cdots c^{(m+1)} y_{n_h}' \end{aligned}$$

for $c^{(m+1)} = c^{(m)} c_{m+1}$.

Conjugating $d_g g d_g^{-1}$ and $d_h h d_h^{-1}$ by y_0 and y_0' respectively, we achieve item (i) of the claim for $m+1$ and thus the result follows. □

As $C < G$ is a BCMS-D subgroup, by the claim above, there is some $n \leq D$, $d_g \in C$, and a CM-sequence $(c_0, \ldots, c_n)$ such that

$$d_g g d_g^{-1} = c^{(n)} z_1 \cdots c^{(n)} z_{n_g}$$

with $z_i \in C_{n+2}$ and $c^{(n)} = c_0 \cdots c_n$. Thus all z_i commute with $c^{(n)}$ and we have

$$d_g g d_g^{-1} = \left(c^{(n)}\right)^{n_g} z$$

with $z = z_1 \cdots z_{n_g}$. Let $x = d_g^{-1} c^{(n)} d_g$ and $c = d_g^{-1} z d_g$. Then $g = x^{n_g} c$ and c commutes with x. By construction we have $|x| = |c_0| = |g_1| = |g|/n_g$ and $|x^{n_g}| = |g|$, thus x is cyclically reduced. This finishes the proof of Proposition 5.21. □

5.6. **Proof of Theorem 5.1.** We use the following reduced form of integral chains to prove Theorem 5.1.

Lemma 5.23. *Let G be a group that splits over a BCMS-D subgroup C. Any integral chain d is equivalent to a chain $d' = d_1 + d_2$ where*

(1) $d_1 = \sum_{i=1}^n g_i$ for some $n \geq 0$, where every g_i is cyclically reduced (see Definitions 2.20 and 2.24) and does not conjugate into any vertex group,
(2) every term of d_2 lies in some vertex group,
(3) there is no $1 \leq i \leq j \leq n$ such that $g_i = g'c$ where g' is a conjugate of g_j^{-1} and $c \in C$ commutes with g',
(4) there is no $1 \leq i \leq n$ such that $g_i = x^m c$ for some $m > 1$, $x \in G$, and $c \in C$ so that x and c commute, and
(5) for every $1 \leq i \leq n$ we have that g_i is CM-reduced (Definition 5.15).

Proof. Given an expression $d' = d_1 + d_2$ of integral chains, where $d_1 = \sum_{j=1}^m k_j h_j$ with cyclically reduced words $h_j \in G$ and $k_j \in \mathbb{Z}_+$, and every term of d_2 lies in some vertex group, associate a complexity $n(d') = \sum_{j=1}^m |h_j|$.

There exists a chain equivalent to d that admits such an expression by replacing elements in d by suitable conjugates so that they are either cyclically reduced or in a vertex group.

Let $d' = d_1 + d_2$ with $d_1 = \sum_{i=1}^n k_i g_i$ be an expression of this form for a chain equivalent to d where $n(d')$ is minimal among such equivalent chains. We claim that d' satisfies the conditions (1)–(4). Each g_i is cyclically reduced by our requirement, and the first two conditions are easy to verify. If there are $1 \leq i \leq j \leq n$ such that $g_i = g'c$ where c commutes with g' and g' is conjugate to g_j^{-1}, then g_i is equivalent to the chain $g' + c$ by (3) of Definition 2.4 and equivalent to $-g_j + c$ by equivalence (1) and (2) of Definition 2.4. Thus we may cancel g_i and g_j at the cost of changing d_2 until one term has coefficient zero to reduce $n(d')$. Similarly we see that if $g_i = x^m c$ where $m > 1$ and c commutes with x, then we may replace $k_i g_i$ by $mk_i x + c$, which has smaller complexity since $|x| < m|x| = |x^m| = |g_i|$.

Finally we can always make the chain d' above further satisfy (5): by Proposition 5.16 we may replace every (cyclically reduced) g_i by $h_i + c_i$ where h_i is CM-reduced, $c_i \in C$ lies in the edge group (and thus in a vertex group). Moreover, Proposition 5.16 shows that g_i is conjugate to $h_i c_i$ by an element of C, so $h_i \in C g_i C$ must be represented by a cyclically reduced word as g_i is, and we have $|h_i| = |g_i|$. This operation does not affect the complexity of the expression and thus the chain d' admits a desired expression. □

We can now prove Theorem 5.1:

Theorem 5.1. *Let G be a graph of groups where each edge group is a BCMS-D subgroup of G. Let c be an integral chain in G. Then either c is equivalent (Definition 2.4) to a chain $\tilde{c}$ such that every term lies in a vertex group or*

$$\mathrm{scl}_G(c) \geq \frac{1}{12(D+2)}.$$

Proof. Fix a CM-subgroup choice $\mathcal{I}(G)$. Assume first that the graph of groups is either an amalgamated free product or an HNN extension over a BCMS-D subgroup C.

Let $c' = c_1 + c_2$ be a chain equivalent to c as in Lemma 5.23 with $c_1 = \sum_{i=1}^n g_i$.

Suppose $n > 0$ and without loss of generality assume that g_1 has the longest length. Set $N = D + 2$ and suppose that

$$\mathrm{scl}_G(c) < \frac{1}{12N}.$$

By Theorem 4.1 there is some $1 \leq j \leq n$ and a cyclic conjugate h of g_j^{-1} such that

$$g^N = h^k h' c,$$

where h' is a prefix of h and $c \in C$. Since $|g_1|$ is maximal among all g_i we conclude that $|g| \geq |h|$. Now consider two cases:

- $|g| = |h|$. Since all of g, h and h' are cyclically reduced, we must have $g^N = h^N c$ in this case. Since g is CM-reduced, by Proposition 5.19 there is some $z \in C$ which commutes with g such that $g = hz$. This contradicts (3) of Lemma 5.23.
- $|g| > |h|$. In this case, Proposition 5.21 implies that there is some $x \in G$, $m \geq 2$ and $c \in C$ such that $g = x^m c$. This contradicts (4) of Lemma 5.23.

Therefore we must have

$$\mathrm{scl}_G(c) \geq \frac{1}{12N} = \frac{1}{12(D+2)},$$

unless c is equivalent to a chain where all terms lie in vertex groups.

When G is a general graph of groups, the chain is supported on a finite subgraph, so we can proceed by induction on the number of edges in the support. At each step, any chosen edge group C splits the group as an amalgamated free product or an HNN extension over C, depending on whether the edge separates the graph. Note that any BCMS-D edge subgroup of G lying in a subgroup H is also a BCMS-D subgroup of H. Thus either at some stage what we have shown above implies the desired gap, or we can keep replacing the chain by equivalent ones supported in subgraphs with strictly smaller number of edges until every term lies in vertex groups. □

6. Gaps for Graph Products of Groups

In this section we apply Theorem 5.1 from the previous section to obtain gap results for graph products. We will use basic notions and properties of graph products in Section 3.

The lower bounds of scl for integral chains depends on the existence of certain induced subgraphs. Let Δ_n be the simplicial graph with vertex set $\mathrm{V}(\Delta_n) = \{v_0, \ldots, v_n\}$ and edge set $\mathrm{E}(\Delta_n) = \{(v_i, v_j) : |i - j| \geq 2\}$. We call this graph the *opposite path of length n*. For any simplicial graph Γ we define

$$\Delta(\Gamma) := \max\{n \mid \Delta_n \text{ is an induced subgraph of } \Gamma\}.$$

The lower bound we establish has size determined by $\Delta(\Gamma)$. The bound applies to all integral chains except for those equivalent (Definition 2.4) to *vertex chains*.

Definition 6.1. A vertex chain is a chain of the form $c = \sum_{v \in V} c_v$, where each c_v is a chain in the vertex group G_v.

Theorem 6.2 (Gaps for Graph Products of Groups)**.** *Let $\mathcal{G}(\Gamma)$ be a graph product and let c be an integral chain of $\mathcal{G}(\Gamma)$. Then either*

$$\mathrm{scl}_{\mathcal{G}(\Gamma)}(c) \geq \frac{1}{12(\Delta(\Gamma)+2)},$$

or one of the following equivalent statements holds:

(i) *c is equivalent (Definition 2.4) to a vertex chain,*
(ii) *the pure factor chain c^{pf} (Definition 3.11) is a vertex chain.*

We will study vertex chains in detail in Section 7. In particular, we prove the following theorem that computes the stable commutator length of a vertex chain $c = \sum_{v \in V} c_v$ in terms of $\mathrm{scl}_{G_v}(c_v)$ and the structure of the defining graph.

Theorem 6.3 (Vertex chains)**.** *Let $\mathcal{G}(\Gamma)$ be a graph product of groups and let $c = \sum_{v \in V(\Gamma)} c_v$ be a vertex chain, where each c_v is a chain in the vertex group G_v. Then $\mathrm{scl}_{\mathcal{G}(\Gamma)}(c)$ can be computed as a linear programming problem if each $\mathrm{scl}_{G_v}(c_v)$ is known, and it is rational if each $\mathrm{scl}_{G_v}(c_v)$ is. Moreover,*

$$\mathrm{scl}_{\mathcal{G}(\Gamma)}(c) \geq \mathrm{scl}_{G_v}(c_v)$$

for any vertex v.

See the end of Section 7.1 for a proof.

Combining with Theorem 6.2, we have:

Corollary 6.4. *Let $G = \mathcal{G}(\Gamma)$ be a graph product of groups over a finite graph Γ, where each vertex group G_v has a spectral gap $C_v > 0$ for integral chains. Then G also has a gap $C = \min\{\frac{1}{12(\Delta(\Gamma)+2)}, C_v\}$ for integral chains.*

In particular, we have a gap theorem for RAAGs and RACGs; see Theorem 6.16.

We can also construct integral chains with small stable commutator length.

Theorem 6.5 (Chains with small stable commutator length)**.** *Let $\mathcal{G}(\Gamma)$ be a graph product of groups and let $\Delta(\Gamma)$ be as above. Then there is an explicit integral chain δ in $\mathcal{G}(\Gamma)$ such that*

$$\frac{1}{12(\Delta(\Gamma)+2)} \leq \mathrm{scl}_{\Gamma}(\delta) \leq \frac{1}{\Delta(\Gamma)}.$$

This shows that the estimate in Theorem 6.2 is accurate up to a scale of 12.

This section is organized as follows. In Section 6.1 we define the canonical CM-subgroup choice in a graph product $\mathcal{G}(\Gamma)$ and show the nice behavior of CM-subgroup sequences with respect to this choice. In Section 6.2, we show that the subgroup $\mathcal{G}(\Lambda)$ associated to any induced subgraph $\Lambda \subset \Gamma$ has BCMS-$\Delta(\Gamma)$. In Section 6.3 we will see that opposite paths are sources of integral chains with small stable commutator length. Then we prove Theorems 6.2 and 6.5 in Section 6.4. In Section 6.5 we deduce the gap results in the special case of RAAGs and RACGs. Finally as applications, we construct groups with interesting scl spectra in Section 6.6.

6.1. **Canonical CM-choice.** Let $\mathcal{G}(\Gamma)$ be a graph product of groups. Every induced subgraph $\Lambda \subset \Gamma$ induces a subgroup $\mathcal{G}(\Lambda) < \mathcal{G}(\Gamma)$. We find nice CM-representatives with respect to such subgroups.

Lemma 6.6. *Let $\Lambda \subset \Gamma$ be an induced subgraph of Γ, let $g \in \mathcal{G}(\Gamma) \setminus \mathcal{G}(\Lambda)$ and let $\bar{g}$ be the element with the shortest length among all elements in $\mathcal{G}(\Lambda) g \mathcal{G}(\Lambda)$. Then*

(1) $\bar{g}$ is a CM-representative, and
(2) the centralizer $Z_{\mathcal{G}(\Lambda)}(\bar{g}) = \mathcal{G}(\Theta)$ where Θ is the induced subgraph of Λ that consists of all vertices of Λ adjacent to all vertices in the support of $\bar{g}$.

Proof. Let $\bar{g}$ be a word of minimal syllable length in $\mathcal{G}(\Lambda) g \mathcal{G}(\Lambda)$. Then $\bar{g}$ is in particular reduced by Lemma 3.1.

Suppose that there are some $h_1, h_2 \in \mathcal{G}(\Lambda)$ such that $\bar{g} h_1 \bar{g}^{-1} = h_2^{-1}$. Then $h_2 \bar{g} h_1 = \bar{g}$. We may assume that h_1, h_2 are written as reduced words. By Lemma 3.1 there are three cases:

- some letter in h_2 merges with another in $\bar{g}$ and commutes with all the letters in between;
- some letter in h_1 merges with another in $\bar{g}$ and commutes with all the letters in between; or
- some letter in h_2 merges with another in h_1 and commutes with all the letters in between.

The first two cases can not occur by our choice of $\bar{g}$ as we can remove the letter that merges with h_1 or h_2 in $\bar{g}$. Thus we should keep having the last case until the word $h_1\bar{g}h_2$ reduces to $\bar{g}$. The process implies that $h_1 = h_2^{-1}$ and both commute with every letter of $\bar{g}$. This shows that $\bar{g}$ is a CM-representative.

The observation above also implies that a reduced word $h \in \mathcal{G}(\Lambda)$ commutes with $\bar{g}$ if and only if every letter in it commutes with all those in $\bar{g}$. This shows $Z_{\mathcal{G}(\Lambda)}(\bar{g}) = \mathcal{G}(\Theta)$ as in (2). □

As the minimal representatives in the double cosets yields nice and controlled centralizers, we always use them as our CM-choice in what follows. It is not important for our purposes but the method above shows that $\bar{g}$ is the unique element of minimal length in $\mathcal{G}(\Lambda)g\mathcal{G}(\Lambda)$.

Definition 6.7 (canonical CM-choice)**.** Let Γ be a simplicial graph and let $\mathcal{G}(\Gamma)$ be a graph product of groups. We define the canonical CM-subgroup choice $\mathcal{I}(\mathcal{G}(\Gamma))$ as follows: For any induced subgraph $\Lambda \subset \Gamma$ and any $g \in \mathcal{G}(\Gamma)$ we choose $\bar{g}$ a CM-representative of g for $\mathcal{G}(\Lambda) \leq \mathcal{G}(\Gamma)$ as an element with the smallest syllable length in $\mathcal{G}(\Lambda)g\mathcal{G}(\Lambda)$. For any other CM-subgroups we choose the CM-representatives arbitrarily.

Note that for any induced subgraph Λ of Γ, item (1) of Lemma 6.6 shows that $\mathcal{G}(\Lambda) \leq \mathcal{G}(\Gamma)$ is a CM-subgroup. Moreover, under the canonical choice all CM-subgroup sequences have the form

$$\mathcal{G}(\Lambda_{n+2}) \leq \mathcal{G}(\Lambda_{n+1}) \leq \cdots \leq \mathcal{G}(\Lambda_1) \leq \mathcal{G}(\Gamma),$$

where $n \geq 0$ and $\Lambda_{n+2} \subset \cdots \subset \Lambda_1 = \Lambda$ is a proper nested sequence of induced subgraphs except that possibly $\Lambda_{n+2} = \Lambda_{n+1}$. Thus either $\mathcal{G}(\Lambda_{n+2}) = \mathcal{G}(\Lambda_{n+1})$ or $\mathcal{G}(\Lambda_{n+2})$ is a proper CM-subgroup by Lemma 6.6.

Thus to show that $\mathcal{G}(\Lambda) \leq \mathcal{G}(\Gamma)$ is a BCMS-D subgroup, we need to control the length of CM-sequences with respect to the canonical choice. This is what we do in the next subsection.

6.2. **The opposite paths Δ_m and lengths of CM-sequences.** Now we find the maximal length of CM-sequences in a given graph product on a graph Γ with respect to the canonical CM-choice. Then we show that the subgroup associated to any induced subgraph of Γ is BCMS-D for $D = \Delta(\Gamma)$.

Recall that for a graph Γ, we define $\Delta(\Gamma)$ to be the largest number $m \in \mathbb{Z}_+$ such that Δ_m is an induced subgraph of Γ. The only graphs where Δ_1 does not embed as an induced subgraph are complete graphs (including the graph with a single vertex). We set $\Delta(\Gamma) = 0$ if Γ is a complete graph. If all Δ_m are induced subgraphs of Γ, then Γ is necessarily infinite, and we set $\Delta(\Gamma) = \infty$.

For example we see that $\Delta(\Delta_m) = m$. Observe also that $\Delta(\Gamma) \leq |\Gamma| - 1$. We will see that $\Delta(\Gamma)$ controls the length of the longest CM-sequence in subgroups of $\mathcal{G}(\Gamma)$ associated to induced subgraphs.

On the one hand, for arbitrary nontrivial vertex groups, a graph product on the graph Δ_n has a CM-subgroup sequence of length $n + 1$. For $n \in \mathbb{Z}_+$ and $i \in \{1, \ldots, n\}$ let Δ_n^i be the induced subgraph of Δ_n with vertex set

$$V(\Delta_n^i) = \{v_i, \ldots, v_n\}.$$

For arbitrary nontrivial elements $g_i \in G_{v_i}$, we have a CM-sequence $(g_0, \ldots, g_n)$ of length $n + 1$, and the associated CM-subgroup sequence is

$$\{e\} \leq \{e\} \leq \mathcal{G}(\Delta_n^n) \leq \cdots \leq \mathcal{G}(\Delta_n^1) \leq \mathcal{G}(\Delta_n).$$

On the other hand, we can find an induced subgraph isomorphic to some Δ_m from a CM-sequence.

Lemma 6.8. *Let Γ_0 be a graph and let $\Gamma_1 \subset \Gamma_0$ be an induced proper subgraph. Fix arbitrary nontrivial vertex groups to form a graph product $\mathcal{G}(\Gamma_0)$. For the canonical CM-choice, let $(c_0, \ldots, c_m)$ be a CM-sequence with respect to $\mathcal{G}(\Gamma_1) < \mathcal{G}(\Gamma_0)$ of length $m + 1$, and let $C_{m+2} \leq \cdots \leq C_0$ be the associated CM-subgroup sequence. Then there is an induced subgraph Δ_m of Γ.*

To prove Lemma 6.8, we first observe some basic relationship between the graphs defining the subgroups C_i and those supporting c_i. Recall that $C_{i+2} = Z_{C_{i+1}}(c_i)$ for all $0 \leq i \leq m$.

Lemma 6.9. *In the setting of Lemma 6.8, there are induced subgraphs $\Gamma_{m+2} \subset \cdots \subset \Gamma_1 \subset \Gamma_0$ such that for each $0 \leq i \leq m$ there is an induced subgraph $\Lambda_i \subset \Gamma_i$ with the following properties:*

(1) Λ_i is the induced subgraph on the support of c_i for all $0 \leq i \leq m$,

(2) *For each* $0 \le i \le m$, Γ_{i+2} *is the induced subgraph consisting of vertices in* Γ_{i+1} *adjacent to all those in* Λ_i,
(3) $C_i = \mathcal{G}(\Gamma_i)$ *for all* $0 \le i \le m+2$,
(4) $\Lambda_i \setminus \Gamma_{i+1} \neq \emptyset$ *for any* $0 \le i \le m$, *and*
(5) $\Lambda_i \subset \Gamma_i \setminus \Gamma_{i+2}$ *for every* $0 \le i \le m$.

Proof. Bullet (3) holds for $i \in \{0,1\}$ by definition. Now we consider $i \ge 2$. Inductively from $i = 2$ to $i = m+2$, we take bullet (1) as the definition of Λ_i, based on which we define Γ_{i+2} as in bullet (2). Then $\Lambda_i \subset \Gamma_i$ and $\Gamma_{i+2} \subset \Gamma_{i+1}$ by definition, and bullet (3) follows from Lemma 6.6. Then bullet (4) holds since $c_i \notin C_{i+1}$ (as a CM-representative).

To see bullet (5) recall that every vertex of Γ_{i+2} is adjacent to all vertices in Λ_i. If a reduced expression of c_i contains a letter in G_v for some $v \in \Gamma_{i+2}$, then we can shuffle it to the end of c_i, contradicting to the choice of c_i. □

Lemma 6.8 follows from the case $i = m$ in Lemma 6.10 below, which is stated in a way to suit its proof by induction. In below, we say the induced subgraph of Γ_0 on a sequence of (distinct) vertices $(v_0, \dots, v_i)$ is isomorphic to Δ_i as *labeled graphs* if v_j and v_k are adjacent in Γ_0 if and only if $|j-k| \ge 2$.

Lemma 6.10. *In the setup of Lemmas 6.8 and Lemma 6.9, for each* $1 \le i \le m$, *there is a sequence of distinct vertices* $V_i = (v_0, \dots, v_i)$ *of* Γ_0 *such that*

- $v_1, \dots, v_i \in \Gamma_{m-i+1}$,
- $v_0 \in \Lambda_{m-i} \setminus \Gamma_{m-i+1}$,

and the induced subgraph of Γ_0 *on* V_i *is isomorphic to* Δ_i *as labeled graphs.*

Proof. We show this lemma by induction on i. First consider the base case $i = 1$. Let u be an arbitrary vertex in $\Lambda_m \setminus \Gamma_{m+1}$, which exists by bullet (4) of Lemma 6.9. There are two possibilities:

- If $\Lambda_{m-1} \cap \Gamma_m = \emptyset$, there is some $v_0 \in \Lambda_{m-1}$ not adjacent to u since $u \notin \Gamma_{m+1}$; See bullet (2) of Lemma 6.9. Then $v_0 \in \Lambda_{m-1} \setminus \Gamma_m = \Lambda_{m-1}$ and $V_1 = (v_0, u)$ satisfies the desired properties.
- If $\Lambda_{m-1} \cap \Gamma_m \neq \emptyset$, write c_{m-1} as a reduced word and let g_{v_1} be the last letter in c_{m-1} that is supported on some $v_1 \in \Gamma_m$. Then there must be some letter g_{v_0} in c_{m-1} supported on $v_0 \in \Lambda_{m-1}$ appearing after g_{v_1} such that v_0 and v_1 are not adjacent, since otherwise we can shuffle g_{v_1} all the way to the end of c_{m-1} contradicting the fact that c_{m-1} has the shortest syllable length in $C_m c_{m-1} C_m$ and $C_m = \mathcal{G}(\Gamma_m)$. Note that $v_0 \notin \Gamma_m$ since g_{v_1} is the last letter on a vertex in Γ_m. Thus $V_1 = (v_0, v_1)$ satisfies the desired properties.

Suppose the lemma holds for some $1 \le i \le m-1$ with a sequence of vertices $V_i = (v_0, \dots, v_i)$. The simplest attempt to obtain V_{i+1} is to add a suitable vertex w_0 at the beginning of V_i. Since $v_0 \notin \Gamma_{m-i+1}$, there is some vertex w_0 in Λ_{m-i-1} that is not adjacent to v_0. Note that $v_0 \notin \Lambda_{m-i-1}$ since otherwise it must be adjacent to all vertices in Γ_{m-i+1} and in particular to v_1, contradicting the induction hypothesis. Combining with $v_s \in \Gamma_{m-i+1}$ for $s \ge 1$ and $\Gamma_{m-i+1} \cap \Lambda_{m-i-1} = \emptyset$ by bullet (5) of Lemma 6.9, all vertices $w_\ell \in \Lambda_{m-i-1}$ we construct below are distinct from those in V_i.

Ideally we would like to choose w_0 above so that it lies in $\Lambda_{m-i-1} \setminus \Gamma_{m-i}$, in which case $V_{i+1} := (w_0, v_0, v_1, \dots, v_i)$ is a desired sequence: Observe that $w_0 \in \Lambda_{m-i-1}$ is adjacent to all $v_1, \dots, v_i \in \Gamma_{m-i+1}$ but not to v_0.

The remaining (harder) case is when every vertex in $\Lambda_{m-i-1} \setminus \Gamma_{m-i}$ is adjacent to v_0. In this case we show the following claim to construct another sequence W of vertices so that the concatenated sequence (W, V) has the desired properties once we cut it down to have exactly $i+1$ vertices by removing some vertices in the tail. The vertices in W are listed in reverse order to reflect the order they appear in the inductive process below.

Claim 6.11. *There is a sequence* $W = (w_k, \dots, w_0)$ *of vertices for some* $k \ge 1$ *such that*

- $w_0, \dots, w_{k-1} \in \Lambda_{m-i-1} \cap \Gamma_{m-i}$,
- $w_k \in \Lambda_{m-i-1} \setminus \Gamma_{m-i}$,
- *the induced subgraph on* W *is isomorphic to* Δ_k *as labeled graphs, that is, for* $0 \le s < t \le k$ *the vertices* w_s *and* w_t *are adjacent in* Γ_0 *if and only if* $|s-t| \ge 2$,
- v_0 *is adjacent to* w_ℓ *iff* $\ell > 0$.

Proof of Claim 6.11. By our assumption, there is some $w_0 \in \Lambda_{m-i-1} \cap \Gamma_{m-i}$ not adjacent to v_0. Choose g_{w_0} to be the last letter on c_{m-i-1} supported on a vertex w_0 with this property. Now inductively we can find letters $g_{w_1}, \ldots, g_{w_k}$ of c_{m-i-1} supported on vertices $w_1, \ldots, w_k \in \Lambda_{m-i-1}$ such that

- for each $1 \le \ell \le k$, g_{w_ℓ} is the last letter on c_{m-i-1} after $g_{w_{\ell-1}}$ such that w_ℓ is not adjacent to $w_{\ell-1}$,
- $w_\ell \in \Lambda_{m-i-1} \cap \Gamma_{m-i}$ for all $\ell < k$, and
- $w_k \in \Lambda_{m-i-1} \setminus \Gamma_{m-i}$.

We are guaranteed to end up with some $w_k \notin \Gamma_{m-i}$: if $w_k \in \Gamma_{m-i}$, g_{w_k} cannot commute with all letters after it on c_{m-i-1} by the minimality of c_{m-i-1}, so we can continue the sequence by adding the last letter $g_{w_{k+1}}$ on c_{m-i-1} after g_{w_k} with the property that w_{k+1} is not adjacent to w_k.

Then by construction $W = (w_k, \ldots, w_0)$ consists of distinct vertices and the corresponding induced subgraph in Γ_0 is isomorphic to Δ_k as labeled graphs. By our choice of w_0, we see w_ℓ is adjacent to v_0 iff $\ell > 0$. This constructs the desired sequence W in Claim 6.11. □

Now we finish the proof of Lemma 6.10. By Claim 6.11, for all $0 \le \ell \le k$, w_ℓ is adjacent to $v_1, \ldots, v_i$ as $w_\ell \in \Lambda_{m-i-1}$ and $v_1, \ldots, v_i \in \Gamma_{m-i+1}$. Then for the concatenated sequence $\widetilde{V}_{i+1} := (W, V)$, its corresponding induced subgraph of Γ_0 is isomorphic to Δ_{i+k+1} as labeled graphs, and all vertices lie in Γ_{m-i} except that the first vertex w_k lies in $\Lambda_{m-i-1} \setminus \Gamma_{m-i}$. Thus by taking the first $i+2$ vertices in the sequence $\widetilde{V}_{i+1}$ as our V_{i+1}, this finishes the inductive proof of Lemma 6.10. □

Now we deduce Lemma 6.8 from Lemma 6.10.

Proof of Lemma 6.8. The case of $i = m$ in Lemma 6.10 implies that the induced subgraph of Γ_0 with vertex set $V_m = (v_0, \ldots, v_m)$ is Δ_m. □

Proposition 6.12. *Let Γ be a simplicial graph where $D := \Delta(\Gamma) < \infty$. Let $\mathcal{G}(\Gamma)$ be a graph product on Γ with arbitrary fixed nontrivial vertex groups. Then for any induced subgraph Λ of Γ, the subgroup $\mathcal{G}(\Lambda)$ is has property BCMS-D.*

Proof. It is enough to check the BCMS-D property using the canonical CM-subgroup choice by Proposition 5.11. As we explained at the end of Section 6.1, it suffices to control the length of CM-sequences. By Lemma 6.8, for any CM-sequence $(c_0, c_1, \cdots, c_m)$, there is an induced subgraph of Γ isomorphic to Δ_m. Thus by definition $D = \Delta(\Gamma) \ge m$. Hence $\mathcal{G}(\Lambda)$ is a BCMS-D subgroup. □

6.3. **Stable commutator length in opposite paths.** Let Δ_m be the opposite path on the vertices $\{v_0, \ldots, v_m\}$ as described above. In this section we will see that for any (nontrivial) vertex groups $(G_v)_{v \in \mathrm{V}(\Delta_m)}$ the associated graph product $\mathcal{G}(\Delta_m)$ has an integral chain with small stable commutator length. Choose a nontrivial element $g_i \in G_{v_i}$ for every vertex v_i of Δ_m. For any $m \ge 2$, define a chain δ_m in $\mathcal{G}(\Delta_m)$ as

$$\delta_m := g_{0,m} - g_{0,m-1} - g_{1,m} + g_{1,m-1}.$$

where $g_{i,j} := g_i \cdots g_j$.

The following computation leads to an upper bound of $\mathrm{scl}(\delta_m)$.

Lemma 6.13. *Given $m \ge 2$ and $0 \le i \le m$, for every $1 \le j \le m - i + 1$ we have*

$$g_{i,m}^j = g_{i,m-1}^j c_j,$$

where c_j is recursively defined as follows: $c_1 = g_m$ and for $1 \le j \le m - i$

$$c_{j+1} := g_{m-j,m-1}^{-1} c_j g_{m-j,m}.$$

Proof. We proceed by induction. For $j = 1$ the result is obvious. Suppose the conclusion holds for some $j \in \{1, \ldots, m-i\}$. Then

$$g_{i,m}^{j+1} = g_{i,m}^j \cdot g_{i,m} = g_{i,m-1}^j c_j g_{i,m}$$

Since c_j commutes with all $g_i, \ldots, g_{m-j-1}$ as it is a product of terms g_k for $k \ge m - j + 1$, we see that

$$\begin{aligned} g_{i,m}^{j+1} &= g_{i,m-1}^j c_j \cdot g_{i,m} \\ &= g_{i,m-1}^j g_{i,m-j-1} c_j g_{m-j,m} \\ &= g_{i,m-1}^{j+1} g_{m-j,m-1}^{-1} c_j g_{m-j,m} \\ &= g_{i,m-1}^{j+1} c_{j+1}. \end{aligned}$$

□

Proposition 6.14. *Let $m \geq 2$ and δ_m be the chain in $\mathcal{G}(\Delta_m)$ defined as above. Then*
$$\frac{1}{12(m+2)} \leq \operatorname{scl}_{\mathcal{G}(\Delta_m)}(\delta_m) \leq \frac{1}{m}.$$

Proof. By Lemma 6.13 we have $g_{i,m}^m = g_{i,m-1}^m c_m$ for $i \in \{0,1\}$, and c_m does not depend on i. Thus by Lemma 2.2 we have
$$\operatorname{scl}(g_{i,m}^m - g_{i,m-1}^m - c_m) \leq \frac{1}{2}$$
for $i \in \{0,1\}$. Therefore,
$$\begin{aligned}\operatorname{scl}(m \cdot \delta_m) &= \operatorname{scl}\left(g_{0,m}^m - g_{0,m-1}^m - g_{1,m}^m + g_{1,m-1}^m\right)\\ &\leq \operatorname{scl}\left((g_{0,m}^m - g_{0,m-1}^m - c_m) - (g_{1,m}^m - g_{1,m-1}^m - c_m)\right) \leq 1\end{aligned}$$
by the triangle inequality. Hence we conclude that
$$\operatorname{scl}(\delta_m) \leq \frac{1}{m}.$$

On the other hand we see that $\Delta(\Delta_m) = m$, and thus by Theorem 6.2 (proved below) we have $\operatorname{scl}(\delta_m) \geq \frac{1}{12(m+2)}$, since δ_m is already a pure factor chain that is not a vertex chain. This finishes the proof. □

6.4. **Proofs of Theorems 6.2 and Theorem 6.5.** We now prove Theorems 6.2 and 6.5.

We first prove the following lemma dealing with an essential part of Theorem 6.2.

Lemma 6.15. *Fix an integer $D \geq 1$. If a graph Γ satisfies $\Delta(\Gamma) \leq D$, then every integral chain c in $\mathcal{G}(\Gamma)$ either has $\operatorname{scl}_{\mathcal{G}(\Gamma)}(c) \geq \frac{1}{12(D+2)}$ or is equivalent to an integral vertex chain.*

Proof. For any integral chain $c = \sum_i g_i$, define its support $\operatorname{supp}(c)$ to be the union of $\operatorname{supp}(g_i)$. Let Λ be the induced subgraph of Γ on $\operatorname{supp}(c)$, which is finite and $\Delta(\Lambda) \leq \Delta(\Gamma)$ by definition. We may reduce the assertion to the case $\Gamma = \Lambda$ as follows. Note that $\operatorname{scl}_{\mathcal{G}(\Lambda)}(c) = \operatorname{scl}_{\mathcal{G}(\Gamma)}(c)$ since $\mathcal{G}(\Lambda)$ is a retract of $\mathcal{G}(\Gamma)$. If c is not equivalent to a vertex chain in $\mathcal{G}(\Gamma)$, neither is it as a chain in $\mathcal{G}(\Lambda)$. Hence it suffices to prove the lemma assuming Γ to be a finite graph.

We proceed by induction on the size $|\Gamma|$. The assertion trivially holds when $|\Gamma| = 1$ since c must be a vertex chain in this case.

Suppose for some $n \geq 1$ the assertion holds for all integral chains c in any graph product $\mathcal{G}(\Gamma)$ with $|\Gamma| \leq n$. Consider an integral chain c in some graph product $\mathcal{G}(\Gamma)$ with $|\Gamma| = n+1$ and $\Delta(\Gamma) \leq D$ such that c is not equivalent to a vertex chain. We need to show
$$\operatorname{scl}_{\mathcal{G}(\Gamma)}(c) \geq \frac{1}{12(D+2)}.$$

Pick any vertex v in Γ. If $\Gamma = \operatorname{St}(v)$, where $\operatorname{St}(v)$ denotes the star of v, then $\mathcal{G}(\Gamma) = G_v \times \mathcal{G}(\operatorname{Lk}(v))$, where $\operatorname{Lk}(v)$ denotes the link of v. Then by Proposition 2.14, c is equivalent to a sum of integral chains $c_v + c'$, where c_v is supported on G_v and c' is supported on $\operatorname{Lk}(v)$. Here c' cannot be equivalent to a vertex chain since c is not. Note that $\Delta(\operatorname{Lk}(v)) \leq \Delta(\Gamma) \leq D$ since $\operatorname{Lk}(v)$ is an induced subgraph of Γ. Thus by the induction hypothesis and Proposition 2.14, we have $\operatorname{scl}_{\mathcal{G}(\Gamma)}(c) \geq \operatorname{scl}_{\mathcal{G}(\operatorname{Lk}(v))}(c') \geq \frac{1}{12(D+2)}$.

Now assume $\Gamma \neq \operatorname{St}(v)$. Then $\mathcal{G}(\Gamma)$ splits non-trivially as an amalgam $\mathcal{G}(\Gamma) = \mathcal{G}(\operatorname{St}(v)) \star_{\mathcal{G}(\operatorname{Lk}(v))} \mathcal{G}(\Gamma \setminus v)$. We know that $\mathcal{G}(\operatorname{Lk}(v)) < \mathcal{G}(\Gamma)$ is a BCMS-$\Delta(\Gamma)$ subgroup by Proposition 6.12. Thus, by Theorem 5.1, it suffices to consider the case where c is equivalent to an integral chain $\tilde{c}$ such that every term of $\tilde{c}$ lies in $\mathcal{G}(\operatorname{St}(v))$ or $\mathcal{G}(\Gamma \setminus v)$. We may again split every term supported on $\mathcal{G}(\operatorname{St}(v))$ into terms in G_v and in $\mathcal{G}(\operatorname{Lk}(v)) < \mathcal{G}(\Gamma \setminus v)$. Thus $\tilde{c}$ and c are equivalent to a chain $c' + c_v$, where c' is supported on $\Gamma \setminus v$ and c_v is supported on G_v. By the monotonicity of scl for the retraction $\mathcal{G}(\Gamma) \to \mathcal{G}(\Gamma \setminus v)$, we deduce that
$$\operatorname{scl}_{\mathcal{G}(\Gamma)}(c) = \operatorname{scl}_{\mathcal{G}(\Gamma)}(c' + c_v) \geq \operatorname{scl}_{\mathcal{G}(\Gamma \setminus v)}(c').$$
As c' is not equivalent to a vertex chain since c is not, we have $\operatorname{scl}_{\mathcal{G}(\Gamma \setminus v)}(c') \geq \frac{1}{12(D+2)}$ by the induction hypothesis. □

Proof of Theorem 6.2. Let c be an integral chain in a graph product $\mathcal{G}(\Gamma)$. By Lemma 6.15, either $\mathrm{scl}_{\mathcal{G}(\Gamma)}(c) \geq \frac{1}{12(\Delta(\Gamma)+2)}$ or c is equivalent to a vertex chain. By Proposition 3.12, c is equivalent to such a vertex chain if and only if c^{pf} is a vertex chain. □

Proof of Theorem 6.5. For any graph Γ and a graph product $\mathcal{G}(\Gamma)$ on Γ, the inclusion $i_m : \mathcal{G}(\Delta_m) \to \mathcal{G}(\Gamma)$ is a retract, where $m = \Delta(\Gamma)$. By Proposition 6.14 there is an integral chain δ_m in $\mathcal{G}(\Delta_m)$ such that

$$\frac{1}{12(m+2)} \leq \mathrm{scl}_{\mathcal{G}(\Delta_m)}(\delta_m) \leq \frac{1}{m}.$$

Since a chain in the retract has the same scl as in the whole group (Proposition 2.6), we conclude that $\delta = i_m(\delta_m)$ has the same property. This concludes the proof. □

6.5. **Applications to right-angled Artin Groups and right-angled Coxeter groups.** Our gap theorems can be simplified in the case of right-angled Artin Groups and right-angled Coxeter groups.

Theorem 6.16 (RAAGs and RACGs)**.** *Let G be the right-angled Artin (or Coxeter) group with defining graph Γ. Then for any integral chain c not equivalent to the zero chain, we have* $\mathrm{scl}_G(c) \geq \frac{1}{12(\Delta(\Gamma)+2)}$.

Proof. Note that any null-homologous chain of the form $\sum_v c_v$ in G is equivalent to the zero chain since each vertex group is abelian, where each c_v is a chain in the vertex group G_v. Thus the result follows from Theorem 6.2. □

By Theorem 6.5, the gap above cannot be uniform in the class of RAAGs, although there is uniform gap 1/2 for elements in RAAGs [Heu19b]. It is natural to ask whether this holds analogously for RACGs.

Question 6.17. *Is there a uniform spectral gap for elements in RACGs?*

Note that there is a uniform gap theorem [CH19, Theorem F] for elements in many graph products, but it does not apply to RACGs because of the existence of 2-torsion. However, we are able to characterize elements in RACGs with zero scl.

Corollary 6.18. *Let G be the right-angled Coxeter group with defining graph Γ. Then For any element $g \in G$, we have either* $\mathrm{scl}_G(g) \geq \frac{1}{12(\Delta(\Gamma)+2)}$ *or* $\mathrm{scl}_G(g) = 0$. *Moreover, the latter case occurs if and only if g is conjugate to g^{-1}, or more precisely, $g = ab$ with $a^2 = id$ and $b^2 = id$.*

Proof. The first assertion directly follows from Theorem 6.16. It also implies that $\mathrm{scl}_G(g) = 0$ if and only if g is equivalent to the zero chain. Hence by Proposition 3.14, we obtain the more explicit characterization of such g. □

One can similarly characterize elements with zero scl in other graph products if elements with zero scl are understood in vertex groups.

We also get a uniform gap for integral chains if we add a hyperbolicity assumption. Since the only hyperbolic RAAGs are free groups, we focus on hyperbolic RACGs below.

Corollary 6.19. *Let $G = \mathrm{C}(\Gamma)$ be a hyperbolic right-angled Coxeter group. Then* $\mathrm{scl}_G(c) \geq \frac{1}{60}$ *for any integral chain not equivalent to the zero chain.*

Proof. It is known by [Mou88] that $\mathrm{C}(\Gamma)$ is hyperbolic if and only if the graph Γ has no induced subgraph isomorphic to the cyclic graph of length 4. Note that the graph Δ_4 contains such an induced subgraph with vertices v_0, v_1, v_3, v_4. Thus $\Delta(\Gamma) \leq 3$ if $\mathrm{C}(\Gamma)$ is hyperbolic. Hence the result follows from Theorem 6.16. □

Based on this, we make the following conjecture.

Conjecture 6.20. *There is a uniform constant $B > 0$ such that any hyperbolic C-special (or A-special, see* [HW08] *for definitions) group has a spectral gap B for integral chains.*

If the conjecture holds true, one can use it and the index formula (Proposition 2.9) to establish effective lower bounds for the index of special subgroups in hyperbolic groups. For instance, it is a well-known theorem that every hyperbolic 3-manifold group contains a finite index subgroup that is special (and hyperbolic) [Ago13], but it is unknown whether the index has a uniform upper bound independent of the manifold. This connection was suggested to us by Danny Calegari and motivated this work on scl of integral chains in RAAGs, but we did not anticipate the spectral gap to be non-uniform.

One can also bound $\Delta(\Gamma)$ in terms of other invariants of the graph Γ.

Corollary 6.21. *If Γ is a simplicial graph where each vertex has valence at most $m \geq 0$, then integral chains in* $\mathrm{A}(\Gamma)$ *and* $\mathrm{C}(\Gamma)$ *have a gap* $\frac{1}{12(m+3)}$.

Proof. Note that in Δ_{m+2} the vertex v_0 is adjacent to $m+1$ vertices $v_2, v_3, \cdots, v_{m+2}$. Thus we must have $\Delta(\Gamma) \leq m+1$. We conclude by Theorem 6.16. □

The dimension of a right-angled Artin (resp. Coxeter) group $\mathrm{A}(\Gamma)$ (resp. $\mathrm{C}(\Gamma)$) associated to some simplicial graph Γ is the largest size of cliques in Γ.

Corollary 6.22. *Any right-angled Artin (resp. Coxeter) group $G = \mathrm{A}(\Gamma)$ (resp. $G = \mathrm{C}(\Gamma)$) of dimension at most d has a gap $\frac{1}{12(2d+1)}$ for integral chains.*

Proof. Note from the definition that Δ_{2d} contains a clique of size $d+1$ with vertices $v_0, v_2, \ldots, v_{2d}$. Thus $\Delta(\Gamma) \leq 2d-1$, and the result follows from Theorem 6.16. □

6.6. **Groups with interesting scl spectra.** Theorem 6.16 implies interesting properties of the spectrum of the infinitely generated right-angled Artin group $\mathrm{A}(\Delta_\infty)$.

Proposition 6.23. *The set of values obtained as scl of integral chains in $\mathrm{A}(\Delta_\infty)$ is dense in $\mathbb{R}_{\geq 0}$, and in particular there is no spectral gap. However, there is a gap $1/2$ for elements in $\mathrm{A}(\Delta_\infty)$.*

Proof. Note that $\mathrm{A}(\Delta_\infty)$ retracts to $\mathrm{A}(\Delta_m)$ for any $m \in \mathbb{Z}_+$. Thus

$$\mathrm{scl}_{\mathrm{A}(\Delta_\infty)}(\delta_m) = \mathrm{scl}_{\mathrm{A}(\Delta_m)}(\delta_m) \in \left[\frac{1}{12(m+2)}, \frac{1}{m}\right].$$

by Proposition 6.14, where δ_m is defined in Section 6.3. Thus we obtain a sequence of integral chains whose scl is positive and converges to 0. Taking integer multiples of such integral chains proves the density. The gap $1/2$ for elements in $\mathrm{A}(\Delta_\infty)$ is shown in [Heu19b]. □

No groups were previously known to have a gap for elements but no gap for integral chains.

With a small modification to the group, we can make scl values of *elements* eventually dense in $\mathbb{R}_{\geq 0}$.

Proposition 6.24. *Let $G = \mathrm{A}(\Delta_\infty) \star F_3$, where F_3 is the free group generated by a, b, c. Then $\mathrm{scl}_G(g) \geq 1/2$ for all $g \neq id \in G$, and the set $\{\mathrm{scl}_G(g) \mid g \in [G,G]\}$ is dense in $[3/2, \infty)$.*

Proof. If $g \neq id$ conjugates into $\mathrm{A}(\Delta_\infty)$ or F_3, the lower bound $1/2$ is known by [Heu19b] and [DH91]. Otherwise, the lower bound $1/2$ follows from [Che18] since both factor groups are torsion-free.

As for the density, recall that the integral chain $\delta_m = g_{0,m} - g_{1,m} - g_{0,m-1} + g_{1,m-1}$ has scl between $1/(12(m+2))$ and $1/m$. Applying Proposition 2.8 to $g = cbag_{0,m}^n a^{-1} g_{1,m}^{-n} b^{-1} g_{0,m-1}^{-n} c^{-1} g_{1,m-1}^n$ for any $n \in \mathbb{Z}_+$, we have

$$\mathrm{scl}_G(g) = \mathrm{scl}_{\mathrm{A}(\Delta_\infty)}(n\delta_m) + \frac{3}{2} \in \left[\frac{3}{2} + \frac{n}{12(m+2)}, \frac{3}{2} + \frac{n}{m}\right].$$

The density follows since m and n are arbitrary positive integers. □

7. Scl of vertex chains

We describe an algorithm to compute scl of vertex chains in Section 7.1. This allows us to relate scl to the *fractional stability number* (fsn) of graphs in Section 7.2. In Section 7.3, we observe and explain the similarity in histograms of scl and fsn on random words and graphs, respectively (Figure 1).

7.1. **Computation by linear programming.** Given any vertex chain c, we will give two linear programming problems (P_c) and (P_c^*) that both compute $\mathrm{scl}(c)$. They are dual to each other and thus yield dual solutions. Moreover, feasible solutions of (P_c) yield quasimorphisms with controlled defects and feasible solutions to (P_c^*) yield admissible surfaces.

To describe the linear programming problems, we introduce the following notion.

Definition 7.1. A *stable measure* on a graph Γ is a list of numbers $\mu = (\mu_v)_{v \in \mathrm{V}}$, one for each vertex, such that

- the sum of μ_v over all vertices in any given clique q of Γ is at most 1;
- $\mu_v \geq 0$ for each v.

A set S of vertices is called *a stable set* if they are pairwise non-adjacent in Γ. Equivalently, each clique contains at most one vertex in S. Thus the indicator function of any stable set is a stable measure.

Given a stable measure μ and a vertex chain c, let

$$|\mu|_c := \sum_{v\in \mathrm{V}} \mu_v \cdot \mathrm{scl}_{G_v}(c_v),$$

which is linear in μ_v. Then maximizing $|\mu|_c$ among stable measures is a linear programming problem (P_c) since the defining properties of a stable measure are linear inequalities in μ_v. Note that the set of stable measures is a compact convex rational polyhedron in $\mathbb{R}^{|\mathrm{V}|}$, and thus the problem (P_c) has an optimal solution at a rational point.

In general, one can replace $\mathrm{scl}_{G_v}(c_v)$ by other weights on vertices, and the corresponding problem is called the *fractional weighted stability number* in graph theory; see [GLS84, Page 333]. Note that the result is rational if the weights are, since the feasible set is a rational polyhedron.

To describe its dual problem (P_c^*), we introduce weighted clique cover.

Definition 7.2. Given a vertex chain c, a *weighted clique cover* with respect to c is a list of real numbers (considered as weights) $y = \{y_q\}$, indexed by the cliques q of Γ such that

- the sum of y_q over all cliques containing any given vertex v is at least $\mathrm{scl}_{G_v}(c_v)$.
- $y_q \geq 0$ for every clique q.

For any weighted clique cover y, let $|y| := \sum y_q$, where the sum is taken over all cliques q of Γ.

Then minimizing $|y|$ over all weighted clique covers with respect to c is the linear programming problem (P_c^*) dual to the problem (P_c). Thus they have the same optimal value by the strong duality theorem of linear programming, explained as follows.

Lemma 7.3. *For any vertex chain c in a graph product $\mathcal{G}(\Gamma)$, we have*

$$\max_{\mu} |\mu|_c = \min_{y} |y|,$$

where the maximization is taken over stable measures μ and the minimization is taken over weighted clique covers y.

Proof. Let $Cl(\Gamma)$ be the set of cliques of Γ. Let M_Γ be the 0-1 matrix where the columns are indexed by the vertices $v \in \mathrm{V}(\Gamma)$ and the rows are indexed by all cliques $q \in Cl(\Gamma)$ such that the (q,v)-entry is 1 if and only if $v \in q$.

Then in matrix form, the problem (P_c) is to maximize $s^T \cdot \mu$ subject to $M_\Gamma \cdot \mu \leq 1_{Cl(\Gamma)}$ and $\mu \geq 0$. Here $1_{Cl(\Gamma)}$ is the vector of 1's of length $|Cl(\Gamma)|$ and s is the vector indexed by $\mathrm{V}(\Gamma)$ with entry $\mathrm{scl}_{G_v}(c_v)$ at vertex v. By the strong duality theorem of linear programming [Sch86, Page 91 (19)] the optimal value agrees with the minimal value of $1^T_{Cl(\Gamma)} \cdot y$ subject to $M^T_\Gamma \cdot y \geq s$ and $y \geq 0$, which is the matrix form of (P_c^*). $\square$

The main result of this subsection is that both (P_c) and (P_c^*) compute $\mathrm{scl}_{\mathcal{G}(\Gamma)}(c)$.

Theorem 7.4. *For any vertex chain c in a graph product $G = \mathcal{G}(\Gamma)$, we have*

$$\mathrm{scl}_G(c) = \max_{\mu} |\mu|_c = \min_{y} |y|,$$

where the maximization is taken over stable measures μ and the minimization is taken over weighted clique covers y.

By Lemma 7.3, to prove Theorem 7.4, it suffices to establish the following two lemmas.

Lemma 7.5. *For any vertex chain c in a graph product $G = \mathcal{G}(\Gamma)$, we have $\mathrm{scl}_G(c) \leq |y|$ for any weighted clique cover y with respect to c.*

Lemma 7.6. *For any vertex chain c in a graph product $G = \mathcal{G}(\Gamma)$, we have $\mathrm{scl}_G(c) \geq |\mu|_c$ for any stable measure μ.*

To prove Lemma 7.5, we first show that scl of a vertex chain is increasing in the coefficients.

Lemma 7.7. *Fix a chain c_v in each vertex group G_v. Given numbers $\lambda_v \geq \lambda'_v \geq 0$ for each vertex v, we have $\mathrm{scl}_G(\sum_v \lambda_v c_v) \geq \mathrm{scl}_G(\sum_v \lambda'_v c_v)$.*

Proof. It suffices to show that scl is non-decreasing in every single λ_u fixing λ_v for all $v \neq u$. Split G as an amalgam $\mathcal{G}(\mathrm{St}(u)) \star_{\mathcal{G}(Lk(u))} \mathcal{G}(\Gamma \setminus \{u\})$. Then we think of the vertex chain $c = \sum_v \lambda_v c_v = \lambda_u c_u + \sum_{v \neq u} \lambda_v c_v$ as a sum of two chains supported on the two factor groups.

By [CH19, Theorem 6.2], we have

$$\mathrm{scl}_G(c) = \inf_d [\mathrm{scl}_{\mathcal{G}(\mathrm{St}(u))}(\lambda_u c_u + d) + \mathrm{scl}_{\mathcal{G}(\Gamma\setminus\{u\})}(-d + \sum_{v\neq u} \lambda_v c_v)],$$

where the infimum is taken over all chains d in $\mathcal{G}(Lk(v))$.

Since $\mathcal{G}(\mathrm{St}(u)) = G_u \times \mathcal{G}(Lk(u))$ is a direct product, by Proposition 2.14 we have $\mathrm{scl}_{\mathcal{G}(\mathrm{St}(u))}(\lambda_u c_u + d) = \max(\lambda_u \mathrm{scl}_{G_u}(c_u), \mathrm{scl}_{\mathcal{G}(\mathrm{Lk}(v))}(d))$, which is clearly non-decreasing in λ_u. Thus $\mathrm{scl}_G(c)$ is non-decreasing in λ_u by the formula above. □

Proof of Lemma 7.5. Recall that each induced subgroup is a retract in a graph product, so scl in the two groups agree for any chain in the subgroup (Proposition 2.6). Without loss of generality, assume $\mathrm{scl}_{G_v}(c_v) > 0$ for each vertex v, as otherwise we may consider the problem on the induced subgroup supported on those vertices with this property. Given a weighted clique cover y, for each clique q, define a vertex chain $d_q = \sum_{v\in q} \frac{y_q}{\mathrm{scl}_{G_v}(c_v)} c_v$. Since $\mathcal{G}(q)$ is the direct product of vertex groups G_v for $v \in q$, by Proposition 2.14, we have

$$\mathrm{scl}_{\mathcal{G}(\Gamma)}(d_q) = \mathrm{scl}_{\mathcal{G}(q)}(d_q) = \max_{v\in q} \mathrm{scl}_{G_v}\left(\frac{y_q}{\mathrm{scl}_{G_v}(c_v)} c_v\right) = y_q.$$

Consider the vertex chain $\sum_q d_q = \sum_v \frac{\sum_{q\ni v} y_q}{\mathrm{scl}_{G_v}(c_v)} c_v$. Note that the coefficient of c_v is $\frac{\sum_{q\ni v} y_q}{\mathrm{scl}_{G_v}(c_v)}$, which is no less than 1, the coefficient of c_v in c, by the definition of weighted clique cover. Thus by Lemma 7.7, we have

$$\mathrm{scl}(c) = \mathrm{scl}(\sum_v c_v) \le \mathrm{scl}(\sum_q d_q) \le \sum_q \mathrm{scl}(d_q) = \sum_q y_q = |y|.$$

□

To prove Lemma 7.6, we construct quasimorphisms and use Bavard's duality.

Given a quasimorphism f_v on each vertex group G_v, we can combine them to obtain a quasimorphism f on the graph product $G = \mathcal{G}(\Gamma)$ as follows.

For each element $g \in G$ with reduced expression $g = g_1 \cdots g_n$, we naturally have a vertex chain $s(g) := \sum_i g_i$. This only depends on g since reduced expressions are unique up to syllable shuffling.

Define $f(c) := \sum_v f_v(c_v)$ for all vertex chains and extend f to $\mathcal{G}(\Gamma)$ by setting

$$f(g) := f(s(g))$$

using the splitting above.

Lemma 7.8. *If each f_v is antisymmetric, then the function f defined above is a quasimorphism on G with defect $D(f) = \sup_q \sum_{v\in q} D(f_v)$, where the supremum is taken over all cliques q of Γ.*

Proof. For each clique q and each vertex $v \in q$, we can find elements $g_v, h_v \in G_v$ with $f_v(g_v) + f_v(h_v) - f_v(g_v h_v)$ arbitrarily close to $D(f_v)$. Then for $g_q := \prod_{v\in q} g_v$ and $h_q := \prod_{v\in q} h_v$, we have

$$f(g_q) + f(h_q) - f(g_q h_q) = \sum_{v\in q} [f_v(g_v) + f_v(h_v) - f_v(g_v h_v)],$$

which can be made arbitrarily close to $\sum_{v\in q} D(f_v)$. This proves the "$\ge$" direction.

For the reversed direction, for any $g, h \in G$, we have reduced expressions $g = g_0 q_g x$ and $h = x^{-1} q_h h_0$ as in Proposition 3.4, where $\mathrm{supp}(q_g) = \mathrm{supp}(q_h) = q = \{v_1, \cdots, v_k\}$ is a clique, $q_g = g_1 \cdots g_k$, $q_h = h_1 \cdots h_k$ with $g_i, h_i \in G_{v_i}$, and gh admits a reduced expression $gh = g_0 q_{gh} h_0$ with $q_{gh} = (g_1 h_1) \ldots (g_k h_k) = q_g q_h$. Since each f_i is antisymmetric, we have $f(x) + f(x^{-1}) = 0$. The definition of f implies that for the reduced expression $g = g_0 q_g x$ we have $f(g) = f(g_0) + f(q_g) + f(x)$ and similarly for h and gh. Hence

$$|f(g) + f(h) - f(gh)| = |f(q_g) + f(q_h) - f(q_{gh})| = \left|\sum_{i=1}^k f_{v_i}(g_i + h_i - g_i h_i)\right| \le \sum_{i=1}^k D(f_{v_i}),$$

where the second inequality uses the formula derived in the first part of the proof. This proves the equality. □

Now we are in a place to prove Lemma 7.6.

Proof of Lemma 7.6. For each vertex v, let ϕ_v be an extremal antisymmetric quasimorphism as in Proposition 2.12 for the chain c_v, i.e. we have $\bar{\phi}_v(c_v) = \mathrm{scl}_{G_v}(c_v)$ and $D(\phi_v) = 1/4$.

Given any stable measure $\mu = (\mu_v)$, let $f_v = \mu_v \cdot \phi_v$ and let f be the quasimorphism obtained as above by combining f_v's. Then for each clique q, we have $\sum_{v\in q} D(f_v) = \frac{1}{4}\sum_{v\in q}\mu_v \le 1/4$ by the definition of stable measures. Thus by Lemma 7.8, we have $D(f) \le 1/4$, and thus $D(\bar{f}) \le 1/2$ by Proposition 2.10. Note that $\bar{f}(g_v) = \mu_v \cdot \bar{\phi}_v(g_v)$ for each $g_v \in G_v$ and similarly for any chain in G_v. By Bavard's duality, we have

$$\mathrm{scl}_G(c) \ge \frac{\bar{f}(c)}{2D(\bar{f})} \ge \bar{f}(c) = \sum_v \mu_v \cdot \bar{\phi}_v(c_v) = \sum_v \mu_v \cdot \mathrm{scl}_{G_v}(c_v) = |\mu|_c.$$

□

Proof of Theorem 7.4. We have $\max_\mu |\mu|_c \le \mathrm{scl}_{\mathcal{G}(\Gamma)}(c) \le \min_y |y|$ by Lemmas 7.5 and 7.6. By Lemma 7.3, we know $\max_x |x|_c = \min_y |y|$, which proves the equality. □

Summarizing the results in this subsection, we give a proof of Theorem 6.3.

Proof of Theorem 6.3. The linear programming problems (P_c) and (P_c^*) both compute $\mathrm{scl}_{\mathcal{G}(\Gamma)}(c)$ by Theorem 7.4. Since the optimal solution of (P_c) is achieved at some rational point, we see that $\mathrm{scl}_{\mathcal{G}(\Gamma)}(c)$ is rational when $\mathrm{scl}_{G_v}(c_v)$ is rational for all v. Finally, by taking $x_v = 1$ and $x_u = 0$ for all $u \neq v$, we obtain a stable measure and from the formulation (P_c) we clearly have

$$\mathrm{scl}_{\mathcal{G}(\Gamma)}(c) \ge \mathrm{scl}_{G_v}(c_v)$$

for each vertex v. □

Theorem 7.4 yields an algorithm to compute stable commutator length of vertex chains. This algorithm has been implemented in Python. The code may be found on the second author's website [1].

7.2. **scl and fractional stability number.** In this section we consider the case where all the vertex terms in the vertex chain have the same stable commutator length. This relates scl to well-studied invariants in graph theory.

To be explicit, we construct for a given graph Γ a graph D_Γ and a chain d_Γ in the right-angled Artin group $\mathrm{A}(D_\Gamma)$, such that $\mathrm{A}(D_\Gamma)$ can be also viewed as a graph product over Γ and d_Γ is a vertex chain where each term has scl 1/2.

Definition 7.9 (Double Graph)**.** For a graph Γ with vertex and edge set $\mathrm{V}(\Gamma)$ and $\mathrm{E}(\Gamma)$, let D_Γ be the graph with vertex and edge set

$$\begin{aligned} \mathrm{V}(D_\Gamma) &= \{\mathrm{a}_v, \mathrm{b}_v \mid v \in \mathrm{V}(\Gamma)\} \text{ and} \\ \mathrm{E}(D_\Gamma) &= \{(\mathrm{a}_v,\mathrm{a}_w), (\mathrm{a}_v,\mathrm{b}_w), (\mathrm{b}_v,\mathrm{a}_w), (\mathrm{b}_v,\mathrm{b}_w) \mid (v,w) \in \mathrm{E}(\Gamma)\}. \end{aligned}$$

Moreover, let $d_\Gamma = \sum_{v\in \mathrm{V}(\Gamma)}[\mathrm{a}_v, \mathrm{b}_v]$ in $\mathrm{A}(D_\Gamma)$. Then D_Γ is called the *double graph* and d_Γ the *double chain* associated to Γ.

Definition 7.10 (Fractional Stability Number)**.** Let Γ be a graph. Then the *fractional stability number* of Γ is defined as

$$\mathrm{fsn}(\Gamma) := \max_\mu \sum_v \mu_v,$$

where the maximum is taken over all stable measures μ.

The fractional stability number of a graph is the *fractional chromatic number* of its opposite graph. This invariant appears more frequently in the literature. For a reference to fractional stability number see [SU11]. The results of the previous section implies:

[1] `https://www.nicolausheuer.com/code.html`

Theorem 7.11 (scl and fsn). *Let* Γ *be a graph and let* D_Γ *and* d_Γ *be the associated double graph and double chain respectively. Then*

$$\mathrm{scl}_{\mathrm{A}(D_\Gamma)}(d_\Gamma) = \frac{1}{2}\mathrm{fsn}(\Gamma),$$

where $\mathrm{fsn}(\Gamma)$ *is the fractional stability number of* Γ.

Proof. Note that for each $v \in \Gamma$, the vertices a_v and b_v are not adjacent and hence $\mathrm{A}(\{\mathrm{a}_v, \mathrm{b}_v\})$ is a free group $F_v = F(\mathrm{a}_v, \mathrm{b}_v)$ of rank two. Also note that a_v and b_v are both (resp. not) adjacent to a_u and b_u if v is (resp. not) adjacent to u. Thus we observe that $\mathrm{A}(D_\Gamma)$ is a graph product over Γ, where the vertex groups are the free groups F_v. In this view, d_Γ is a vertex chain where each vertex term is $[\mathrm{a}_v, \mathrm{b}_v]$, which satisfies $\mathrm{scl}_{F_v}([\mathrm{a}_v, \mathrm{b}_v]) = 1/2$. Thus the result follows from Theorem 7.4. □

For the rest of this subsection, we apply known results of fsn on graphs to deduce properties of scl in such groups. We first describe the full spectrum of fsn on graphs. Note that the full spectrum of scl is not known even in the best understood case of free groups.

Proposition 7.12 (see also [SU11, Proposition 3.2.2]). *The set of numbers that appear as* $\mathrm{fsn}(\Gamma)$ *for some nonempty graph* Γ *is*

$$\{1\} \cup [2, \infty) \cap \mathbb{Q}.$$

Proof. We already know that $\mathrm{fsn}(\Gamma)$ is always rational since the feasible set is a rational polyhedron. It is also easy to notice that $\mathrm{fsn}(\Gamma) \geq 1$ since each vertex is a stable set, and that $\mathrm{fsn}(\Gamma) \geq 2$ whenever there are two non-adjacent vertices.

So it suffices to construct graphs to achieve all rational numbers $r \geq 2$. For any $m \geq 2$ and $n \geq 2m$, let $\Gamma_{m,n}$ be the graph with n vertices $v_1, \ldots, v_n$ such that it is the union of cliques on $v_{i+1}, \ldots, v_{i+m}$ for all $1 \leq i \leq n$, where indices are taken mod n. We claim that $\mathrm{fsn}(\Gamma_{m,n}) = n/m$, from which the result would follow.

Using $n \geq 2m$, it is straightforward to check that the cliques used to described $\Gamma_{m,n}$ are all the maximal cliques. Thus having weight $1/m$ on all vertices is a stable measure, which shows $\mathrm{fsn}(\Gamma_{m,n}) \geq n/m$.

On the other hand, assigning weight $1/m$ to each maximal clique (and 0 to all smaller cliques) is a weighted clique cover, and hence $\mathrm{fsn}(\Gamma_{m,n}) \leq n/m$ by the dual problem. Thus $\mathrm{fsn}(\Gamma_{m,n}) = n/m$. □

For comparison, it is known that scl in free groups has a sharp lower bound 1/2, and based on experiments, the spectrum seems to be proper in $[1/2, 3/4)$ and dense in $[3/4, \infty)$. However, it appears to be much harder if possible at all to construct families of elements or integral chains in free groups with scl achieving arbitrary rational numbers greater than 1.

Combining Theorem 7.11 and Proposition 7.12 we deduce:

Theorem 7.13 (Rational realization). *For every rational number* $q \geq 1$ *there is an integral chain* c *in a right-angled Artin group* $\mathrm{A}(\Gamma)$ *such that* $\mathrm{scl}_{\mathrm{A}(\Gamma)}(c) = q$.

Computing the fractional stability number is NP-hard [GLS81]. This implies that computing scl in RAAGs is also NP-hard.

Theorem 7.14 (NP-hardness). *Unless* $P = NP$, *there is no algorithm which, given a simplicial graph* Γ, *an element* $g \in \mathrm{A}(\Gamma)$ *and a rational number* $q \in \mathbb{Q}^+$ *decides if* $\mathrm{scl}_{\mathrm{A}(\Gamma)}(g) \leq q$ *in polynomial time in* $|\mathrm{V}(\Gamma)| + |g|$. *The same holds for chains.*

Proof. It is known [GLS81] that computing fsn for a graph Γ is NP-hard. Given a graph Γ, we may in polynomial time construct the double graph and the double chain $d_\Gamma \in \mathrm{A}(D_\Gamma)$. By Theorem 7.11, computing $\mathrm{scl}(d_\Gamma) = \frac{1}{2}\mathrm{fsn}(\Gamma)$ is NP-hard as well.

Let $\tilde{D}_\Gamma$ be the graph obtained from D_Γ by adding $|V(\Gamma)|$ isolated vertices. Then $\mathrm{A}(\tilde{D}_\Gamma)$ is a free product $\mathrm{A}(D_\Gamma) \star F_{|V(\Gamma)|}$. Using Proposition 2.8, we may in polynomial time construct an element $\tilde{d}$ in $\mathrm{A}(\tilde{D}_\Gamma)$ such that $\mathrm{scl}(\tilde{d}) = \mathrm{scl}(d_\Gamma) + \frac{\mathrm{V}(\Gamma)-1}{2}$. Thus computing scl of elements in RAAGs is also NP-hard. □

7.3. **Histograms of scl and fsn.** Although it is NP-hard, we may compute fsn relatively quickly for graphs with up to 30 vertices. This allows us to perform computer experiments on the distribution of fsn for random graphs. The result of these experiments is recorded (rescaled by 1/2) in Figure 1b in the introduction. Here we considered $50,000$ random graphs with 25 vertices, where between every two vertices there is an edge with probability 1/2. This reveals an interesting distribution of fsn on random graphs: Values with low denominator appear much more frequently and the histogram exhibits a self-similar behavior.

The same type of histogram has been observed for stable commutator length of random elements in the free group (Figure 1a). Here we consider $50,000$ uniformly chosen random words of length 24 in the commutator subgroup of the free group F_2. See [Cal09a, Section 4.1.9] for a discussion of this phenomenon and comparison to Arnold's tongue. Explanations of these patterns in the frequency for either scl or fsn are not known.

In this section we will give a brief statistical analysis of both scl and fsn. We show that both scl and fsn can be modeled using the same type of distributions which we describe in Definition 7.15. While this is purely heuristic, it indicates that fsn and scl converge for large graph sizes / word lengths to a similar distribution; see Question 7.16.

Let SCL denote the random variable $2\cdot\mathrm{scl}(W)$ where W is the random variable with a uniform distribution on $\{w \in F_2 \mid |w| \leq 24\}$ and let FSN be the random variable $\mathrm{fsn}(\Gamma)$ where Γ is a random variable with uniform distribution $\{\Gamma \mid \mathrm{V}(\Gamma) = 25\}$. We note that the factor of 2 for scl is intended and indeed necessary. In light of the relationship to Euler characteristic (Definition 2.1) and Bavard's Duality Theorem (Theorem 2.11), $2\cdot\mathrm{scl}$ seems to be the more natural invariant. The histograms of $50,000$ independent instances of SCL and FSN may be found in Figure 7.

We make two crucial heuristic observations:

(1) For large integers n, we observe that $\mathbb{P}(X \text{ has denominator n}) \sim \frac{\phi(n)}{n^d}$, where ϕ is Euler's Totient function and X is SCL or FSN. This is depicted in Figure 6a. Experimentally we may estimate that $d \sim 1.7$ for SCL and $d \sim 2.5$ for FSN. It is also apparent that for smaller n this heuristic does not hold, and that instead this coefficient is much smaller. This suggests that the exponent may be approximated by $d \cdot (1 - n^{\beta})$ for some negative β.
(2) For a fixed denominator n, let X_n be the random variable of X conditioned on that X has denominator n. Then X_n follows roughly a normal distribution B_n (rounded to the closest rational in $1/n$) with fixed mean μ and standard deviation σ_n; see Figure 6b. The standard deviation appears to be roughly of the form $\sigma_n = c_1 \cdot n^{c_2}$; see Figure 6c.

This suggests that both the histogram of scl and fsn are the result of an interference of several (rounded) 'normal' distributions B_n.

These observations lead us to the following construction of a random variable X depending on real parameters d, β, μ, c_1, c_2.

Definition 7.15 (The distribution X)**.** Let $d < -1$, $\beta < 0$, $c_1 > 0$, $c_2 < 0$, and μ be real parameters. Define the random variable $X = X(d, \beta, \mu, c_1, c_2)$ as follows:

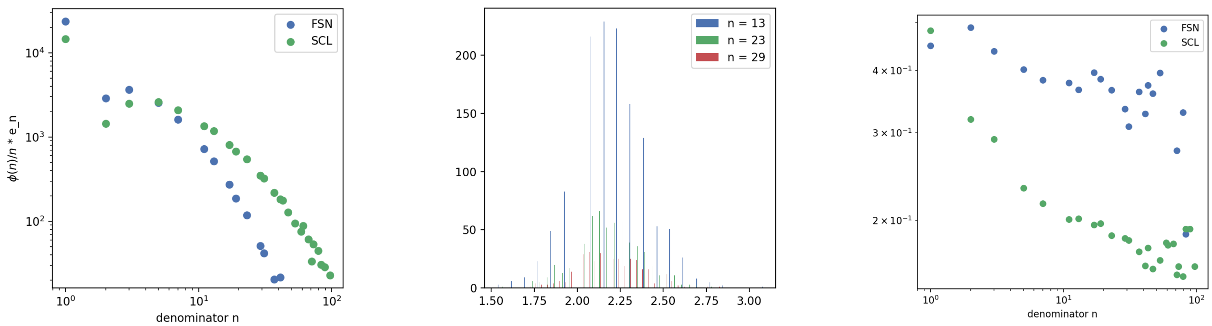
Set $p(n, \beta, d) = n^{(1-n^{\beta})\cdot d}$. Choose with probability $p(n, \beta, d)/\sum_{n=1}^{\infty} p(n, \beta, d)$ an integer $n \in \mathbb{N}$. Choose the rational X in $\frac{1}{n}\mathbb{Z}$ as follows: Let N_n be the random variable with distribution $\mathcal{N}(\mu, (c_1 \cdot n^{c_2})^2)$, the normal distribution with mean μ and standard deviation $c_1 \cdot n^{c_2}$. Set X to be the number in $\frac{1}{n}\mathbb{Z}$ closest to N_n.

The distribution of X may be found on the second authors website [2]. We may use this to fit X to SCL and FSN. The result of this experiment is shown in Figure 7. At least qualitatively, X is a good approximation of the distribution of SCL and FSN.

Based on this, we ask:

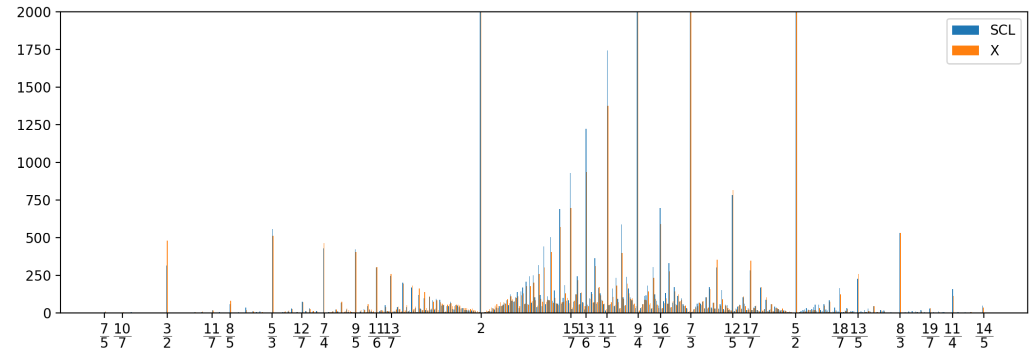
Question 7.16. *Is there a* natural *distribution Y indexed by some parameter set $\mathcal{P}$ such that there are sequences of parameters s_n, f_n for $n \in \mathbb{N}$ such that as $n \to \infty$, both the random variable* $\mathrm{scl}(w)$, *for w uniformly chosen from $\{w \in [F_2, F_2] \mid |w| = 2 \cdot n\}$, and* $\mathrm{fsn}(\Gamma)$ *where Γ is uniformly chosen among all graphs with n vertices converge almost surely to $Y(s_n)$ and $Y(f_n)$, respectively?*

[2] `https://www.nicolausheuer.com/code.html`

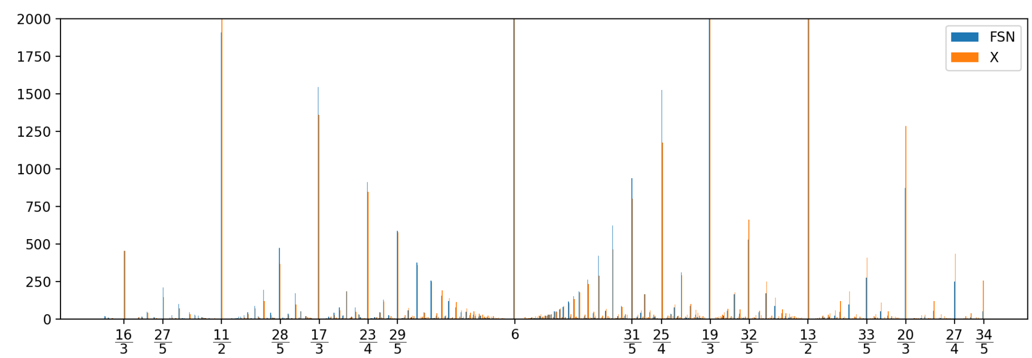


(A) (B) (C)

FIGURE 6. Statistical analysis of SCL and FSN: Let X be either a random scl in F_2 on words of length 24 or a fsn of a random graph on 25 vertices. Let X_n denote the set of elements with denominator exactly n. Figure 6a plots $e_n = \#X_n$, the number of elements having denominator n for $50,000$ random samples. Figure 6b shows the distribution of SCL having denominator 13, 23 and 29 for $50,000$ samples. Figure 6c shows the different standard sample deviations of X_n.



(A) $2 \cdot \mathrm{scl}(w)$ for $w \in F_2$ in the commutator subgroup with length 24 uniformly chosen for $50,000$ instances (green) vs. $50,000$ random instances of the X distribution modeled with parameters $d = -2$, $\beta = -0.2$, $\mu = 2.164$, $c_1 = 0.3$ and $c_2 = -0.14$ (blue)



(B) $\mathrm{fsn}(\Gamma)$ for Γ uniformly chosen as a graph with 25 vertices for $50,000$ instances (green) vs. $50,000$ random instances of the X distribution with parameters $d = -2.8$, $\beta = -0.2$, $\mu = 6.141$, $c_1 = 0.5$ and $c_2 = -0.1$ (blue).

FIGURE 7. Modeling $2 \cdot \mathrm{scl}$ and fsn using the X distribution. In both cases, we truncated the spikes to fit the figure.

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