TOPIC PROPOSAL: STABLE COMMUTATOR LENGTH

LVZHOU CHEN
ADVISOR: DANNY CALEGARI

Contents

1. Introduction

Associated to each group, there is a canonical function scl called the stable commutator length, which can be thought as a topological extremal problem about surface mappings or interpreted as a relative version of Gromov's norm as we will see below. The interesting question is how the behavior of scl is related to other properties of the group, but it is hard to get a good answer right now because people do not know how to compute it until recent years. So the current study of scl is more about how to compute scl and how it behaves on certain groups. We will get better answers to that question when we have a clearer picture for behaviors of scl on different kinds of groups. Besides, the study of scl is quite interesting itself, since it is related to surface mappings into spaces and linear programming, quasimorphisms and bounded cohomology, and dynamics.

2. Group Theoretic Definition of scl

In this section, we give the group theoretic definition of scl and some basic examples.

Definition 2.1. For any group G and any $g \in [G, G]$, the commutator length of g, denoted $\mathrm{cl}_G(g)$, is the minimal integer n that we can write $g = \prod_{k=1}^{n}[a_k, b_k]$ for some $a_k, b_k \in G$. Then define the stable commutator length of g to be

$$\mathrm{scl}_G(g) = \lim_{m\to\infty} \frac{\mathrm{cl}_G(g^m)}{m},$$

where the limit always exists since $\{\mathrm{cl}_G(g^m)\}_{m=1}^{\infty}$ is a sub-additive sequence. We simply write cl and scl when the group G is clear from the context.

Date: Feb. 4, 2016. Updated on May 11, 2016.

From the geometric group theory point of view, the subgroup $[G, G]$ has a canonical generating set, namely the set of commutators, which induces a canonical word metric d, then $\mathrm{cl}(g) = d(g, 1)$ and $\mathrm{scl}(g)$ can be thought as "the growth of d in direction g". One advantage of taking the stabilized version is suggested by the following

Example 2.2. scl is homogeneous by definition, i.e. $\mathrm{scl}(g^k) = k\mathrm{scl}(g)$ for any $k \geq 1$. But cl is not. For example, fix $G = F_2$ to be the free group generated by a, b, then $\mathrm{cl}([a, b]) = 1$, $\mathrm{cl}([a, b]^2) = 2$, while $\mathrm{cl}([a, b]^3) = 2$ since $[a, b]^3 = [aba^{-1}, b^{-1}aba^{-2}][b^{-1}ab, b^2]$.

Some basic but important properties of cl and scl follows easily from the definition.

Proposition 2.3. *(1)* cl *and* scl *are monotone: Let $f : G \to H$ be a group homomorphism, then $\mathrm{cl}_H(f(g)) \leq \mathrm{cl}_G(g)$ and $\mathrm{scl}_H(f(g)) \leq \mathrm{scl}_G(g)$ for any $g \in [G, G]$.*
(2) cl *and* scl *are characteristic: If $f : G \to G$ is an isomorphism, then $\mathrm{cl}_G(f(g)) = \mathrm{cl}_G(g)$ and $\mathrm{scl}_G(f(g)) = \mathrm{scl}_G(g)$ for any $g \in [G, G]$. In particular,* cl *and* scl *are class functions.*

Recall from group homology that $B_1(G) = B_1(G; \mathbb{R})$ consists of homologically trivial real 1-chains and contains $[G, G]$. One can extend scl to $B_1(G)$.

Definition 2.4. Let $g_1, \ldots, g_n \in G$ such that $\prod_{i=1}^n g_i \in [G, G]$, define

$$\mathrm{cl}(g_1 + g_2 + \ldots + g_n) = \inf_{\{t_1,\ldots,t_n\}} \mathrm{cl}(\prod_{i=1}^{n} t_i g_i t_i^{-1})$$

and

$$\mathrm{scl}(g_1 + g_2 + \ldots + g_n) = \lim_{m\to\infty} \mathrm{cl}(g_1^m + \ldots + g_n^m).$$

Notice that $\mathrm{scl}(c+g+g^{-1}) = \mathrm{scl}(c)$ where $c = g_1+g_2+\ldots+g_n$ as above, thus we can regard $-g$ as g^{-1} in an integral 1-chain. This defines scl on homologically trivial integral 1-chains. Further notice that scl is homogeneous and $\mathrm{scl}(c_1 + c_2) \leq \mathrm{scl}(c_1) + \mathrm{scl}(c_2)$, therefore scl uniquely extends to a pseudo-norm on $B_1(G)$.

Proposition 2.3 also holds for chains.

Example 2.5. It is immediate from the definition that scl vanishes when cl is bounded, which is obviously true for finite groups, uniformly perfect groups, and abelian groups.

In contrast, cl is complicated even on finite groups. For example, a theorem due to Liebeck-O'Brien-Shalev-Tiep [19] shows that $\mathrm{cl}(g) = 1$ for any $g \neq 1$ in a nonabelian finite simple group, which is far from obvious. Thus a potential advantage of working with scl instead of cl is to get rid of some crazily complicated bounded error as in geometric group theory. The algebraic definition of scl, however, looks quite complicated and the calculation seemingly requires knowledge about cl. Therefore, it is natural to seek for descriptions of scl without mentioning cl if one wishes to know scl without knowing much about cl. This is what we do in the sequel.

3. scl and Surface Mappings

In this section we describe scl as a topological extremal problem, namely finding how efficient one can bound a given collection of loops in a space by surfaces, and explain how the computation of scl on free groups and more generally on free products can be done by reducing to linear programmings.

First, we introduce the following standard notation.

Definition 3.1. Let S be a compact surface. Define

$$\chi^-(S) = \sum_i \min(0, \chi(S_i))$$

where S_i are the components of S and χ is the Euler characteristic. Equivalently, $\chi^-(S)$ is the Euler characteristic of S after removing disk and sphere components.

The following proposition relates scl to surface mappings, and it basically follows from taking covers and cut-and-paste of surfaces.

Proposition 3.2 (Calegari, [9])**.** *Let $g_i \in G$ ($1 \le i \le k$) such that their product is in $[G,G]$. Realize G as $\pi_1(X)$ for some space X. Let $\gamma_i : S^1 \to X$ represent the conjugacy class of g_i. A compact oriented surface S (not necessarily connected) together with a map $f : S \to X$ is called admissible of degree $n(S) \ge 1$ if the following diagram commutes*

$$\begin{tikzcd} \partial S \arrow[r, "i"] \arrow[d, "\partial f"'] & S \arrow[d, "f"] \\ \bigsqcup_i S^1 \arrow[r, "\sqcup \gamma_i"] & X \end{tikzcd}$$

where i is the inclusion and $\partial f_*[\partial S] = n(S)[\sqcup S^1]$.

Then

$$\mathrm{scl}(g_1 + g_2 + \cdots + g_k) = \inf_S \frac{-\chi^-(S)}{n(S)}$$

where the infimum is taken over all admissible surfaces.

The quantity $\frac{-\chi^-(S)}{n(S)}$ measures a sort of efficiency of the surface map, and note that it is invariant under taking a finite cover of S. A priori there may not exist a most efficient admissible surface, but it would be very special if exists.

Definition 3.3. An admissible surface is *extremal* if it achieves the infimum.

Proposition 3.4 (Calegari, [9])**.** *Extremal surfaces are π_1-injective.*

Now let $H(G)$ be the subspace of $B_1(G)$ spanned by elements of the form $ng - g^n$ and $g - hgh^{-1}$, then it is easy to see from the proposition above that scl vanishes on $H(G)$ and thus descends to a pseudo-norm on the quotient. We will see in next section that scl is a genuine norm on the quotient if G is word hyperbolic (Theorem 4.32).

Definition 3.5. Define $B_1^H(G) = B_1(G)/H(G)$. We say a group homomorphism $f : G_1 \to G_2$ is an *isometric embedding* if f is injective and the induced map $f : B_1^H(G_1) \to B_1^H(G_2)$ preserves scl, i.e. $\mathrm{scl}_{G_1}(c) = \mathrm{scl}_{G_2}(f(c))$ for all $c \in B_1^H(G_1)$.

The most immediate example of isometric embedding comes from retraction:

Example 3.6. Let $i : H \to G$ and $r : G \to H$ be group homomorphisms such that $r \circ i = \mathrm{id}_H$, then i is an isometric embedding by Proposition 2.3. In particular, calculation of scl on free product of infinitely many groups reduces to free product of finitely many groups.

3.1. **scl on Free Groups.** So far, we have not given any group G where scl does not vanish and we can compute it precisely. A milestone of the study of scl is the following theorem (mainly) by Danny Calegari about scl on free groups. Note that calculation of scl on free group would give universal upper bound of scl on any other group by the universality of free groups and the monotonicity of scl.

Theorem 3.7. *Let G be the free group of rank $n \geq 2$, then:*
(1)(Calegari, [7])scl *is piecewise rational linear (see definition below) on $B_1^H(G)$, in particular, it takes rational values on $[G,G]$.*
(2)(Calegari, [7])*Every nonzero rational chain in $B_1^H(G)$ rationally bounds an extremal surface.*
(3)(Calegari, [7])*The computation of* scl *on $B_1^H(G)$ can be done precisely by reducing to linear programming problems. The program* `scallop` *(*[13]*) and* `wallop` *(*[22]*) computes* scl *and finds extremal surfaces respectively.*
(4)(Calegari-Walker, [12])*The image of $[G,G]$ under* scl *contains elements congruent to every element of $\mathbb{Q}$ mod $\mathbb{Z}$. Moreover, it contains a well-ordered sequence of values with ordinal type ω^ω.*
(5)(Duncan-Howie, [14])*There is a spectral gap:* $\mathrm{scl}(g) \geq 1/2$ *for all* $g \neq 1 \in [G,G]$.

Definition 3.8. We say scl is *piecewise rational linear* if it is piecewise rational linear on every finite dimensional rational subspace of $B_1^H(G)$.

Remark 3.9. It is very amazing that scl takes rational values since by definition it is a limit (or an infimum resp.) over a sequence of (or infinitely many resp.) rational numbers, which a priori could be irrational. Actually there are finitely presented groups where scl takes irrational values ([23]).

One can do computer experiments using the program to calculate scl on free groups to see how scl behaves on free groups. Here are several questions about behavior of scl on free groups with conjectural answers based on experiments:

Question 3.10. Let G be the free group of rank $n \geq 2$,
(1)(Calegari)does the closure of the image of $[G,G]$ under scl contain any interval? (Yes?)
(2)(Calegari)does the spectral gap 1/2 as in Theorem 3.7 (5) hold for nonzero integral chains? (Yes?) In particular, is $\mathrm{scl}(x+y+x^{-1}y^{-1}) = 1/2$ true whenever the chain $x+y+x^{-1}y^{-1}$ is non-trivial? (Yes?)
(3)(Calegari-Walker [12])Does every injective homomorphism $f: F_2 \to G$ induce an isometric embedding? (Yes?)

3.2. **scl on Free Products.** Theorem 3.7 and the conjectural phenomena suggested in questions above more or less give the rough picture we have so far for scl on free groups. One can expect that scl behaves similarly on generalizations of free groups, such as free products and word hyperbolic groups. We focus on free products here and defer the discussion of scl on hyperbolic groups to the next section.

Inspired by John Stalling, Danny Calegari ([8]) studies surface mappings into free products of (free) abelian groups and shows that the calculation of scl again can be solved by linear programming and is piecewise rational linear. Moreover, the argument here can be used to study scl on families of elements. In the same paper it is shown that (2) in Theorem 3.7 is a special phenomenon for free groups:

Example 3.11 (Calegari, [8])**.** Let $G = \mathbb{Z} * \mathbb{Z}^2$, where the first factor is generated by a and the second is by v_1, v_2, then the chain $c = av_1^2a^{-1}v_1^{-1} + v_2 + v_1^{-1}v_2^{-1}$ has $\mathrm{scl}(c) = 1/2$ but does not rationally bound any extremal surface.

Interestingly, it turns out that the argument in ([8]) can be improved to get a glimpse of how scl behaves in general free products, and everything except (2) in Theorem 3.7 can be generalized.

Theorem 3.12 (Chen, in preparation)**.** *Let $G = *_\lambda G_\lambda$ be a free product of at least 2 nontrivial groups, then:*
*(1)*scl *is piecewise rational linear on $B_1^H(G)$ if* scl *vanishes on each factor G_λ.*
(3)The computation of scl *on $B_1^H(G)$ reduces to linear programming problems if* scl *vanishes on each factor G_λ. It can be computed in practice if one has a good knowledge on solving the word problem in each G_λ.*
(4)The image of $[G,G]$ under scl *contains elements congruent to every element of $\mathbb{Q}$ mod $\mathbb{Z}$. Moreover, it contains a well-ordered sequence of values with ordinal type ω^ω*
(5)There is a spectral gap $1/2$ if each G_λ is torsion free: $\mathrm{scl}(g) \geq 1/2$ *for all $g \in [G,G]$ that does not conjugate to an element in some G_λ.*

Remark 3.13. (a)For part (1) in the theorem above, the best one can hope is that the conclusion holds when scl is piecewise rational linear on each factor G_λ. We do not know whether this is true (maybe not).
(b)scl vanishes on many of groups such as amenable groups, SL(n,$\mathbb{Z}$) ($n \geq 3$), $\mathrm{Homeo}^+(S^1)$ and subgroups of $\mathrm{PL}^+(I)$.
(c)Part (5) is really about giving uniform upper bound of a certain kind of concave piecewise linear functions. One may hope that a similar analysis could solve Question 3.10 (2). One can get a gap smaller than 1/2 if G_λ has torsion.
(d)Part (4) follows from the free product case and the following isometric embedding theorem, and the proof of which is along the same line as that of the theorem above.

Theorem 3.14 (Chen, in preparation)**.** *Let $f_\lambda : A_\lambda \to B_\lambda$ be a family of isometric embeddings with respect to* scl*, then the induced map $f : *A_\lambda \to *B_\lambda$ is also an isometric embedding.*

Corollary 3.15. Let $g_\lambda \in G_\lambda$, $G = *_\lambda G_\lambda$, and $f_\lambda : \langle g_\lambda \rangle \to G_\lambda$ be the inclusion. Then the induced map $f : *_\lambda \langle g_\lambda \rangle \to G$ is an isometric embedding. In particular, let $H = *_\lambda(\mathbb{Z}/\mathbb{Z}_{k_\lambda})$, then the image of $[G,G]$ under scl_G contains that of [H,H] under scl_H if G_λ has an element of order k_λ ($k_\lambda \geq 2$ could be ∞, in which case $\mathbb{Z}/\mathbb{Z}_{k_\lambda}$ refers to $\mathbb{Z}$).

It is natural to ask how scl behaves on amalgams (or more generally graphs of groups) instead of free products. The only result in this direction known so far is:

Theorem 3.16 (Susse, [20])**.** *Let $G = A *_{\mathbb{Z}^k} B$ where A and B are free abelian groups of rank at least $k \geq 1$, then* scl *is piecewise rational linear on $B_1^H(G)$ and can be calculated by computer algorithm.*

In particular, it is not known how scl behaves on the fundamental groups of closed surfaces. Free groups are the fundamental groups of open surfaces, thus one may expect that scl behaves similarly (e.g. piecewise rational linear) on surface groups.

4. scl and Quasimorphisms

In this section we relate scl to quasimorphisms which is closely related to bounded cohomology.

Many objects in mathematics can be studied by functions on it. For groups, homomorphisms to $\mathbb{R}$ are rare and only see the abelianization. Quasimorphisms generalize homomorphisms to $\mathbb{R}$ and are very rich in many cases.

Definition 4.1. Let G be a group, a map $\phi : G \to \mathbb{R}$ is a *quasimorphism* if

$$D(\phi) := \sup_{g,h\in G} |\phi(g) + \phi(h) - \phi(gh)| < \infty,$$

and $D(\phi)$ is called the *defect* of ϕ. Moreover, a quasimorphism ϕ is called *homogeneous* if $\phi(g^n) = n\phi(g)$ for any $n \in \mathbb{Z}$ and $g \in G$. Denote the vector space of quasimorphisms and homogeneous quasimorphisms by $\widehat{Q}(G)$ and $Q(G)$ respectively. Note that $H^1(G;\mathbb{R})$, the space of homomorphisms to $\mathbb{R}$, is the subspace of quasimorphisms with defect 0 in $Q(G) \subset \widehat{Q}(G)$.

For homogeneous quasimorphism, it is easier to find the defect:

Proposition 4.2 (Bavard, [4])**.** *Let $\phi \in Q(G)$, then*

$$D(\phi) = \sup_{(a,b)\in G^2} \phi([a,b])$$

Any quasimorphism is close to a (unique) homogeneous one.

Proposition 4.3. *For any $\phi \in \widehat{Q}(G)$, there is a unique $\bar{\phi} \in Q(G)$, called the homogenization of ϕ, such that $\phi - \bar{\phi}$ is bounded. Explicitly, for any $g \in G$,*

$$\bar{\phi}(g) = \lim_{n\to\infty} \frac{\phi(g^n)}{n} = \inf_{n\geq 1} \frac{\phi(g^n) + D(\phi)}{n} = \sup_{n\geq 1} \frac{\phi(g^n) - D(\phi)}{n}$$

and $D(\bar{\phi}) \leq 2D(\phi)$.

Example 4.4 (Barge-Ghys, [2])**.** Let (M, g) be a closed hyperbolic manifold, α be a 1-form on M and p be a fixed base point on M. For any $\gamma \in \pi_1(M)$, it is uniquely represented by a geodesic arc L_γ with both end points p, and corresponds to a unique geodesic loop $\bar{L}_\gamma$ homotopic to L_γ. Define $\phi_{p,\alpha}(\gamma) = \int_{L_\gamma} \alpha$ and $\phi_\alpha(\gamma) = \int_{\bar{L}_\gamma} \alpha$. Then $\phi_{p,\alpha}$ is a quasimorphism by Stokes theorem since $d\alpha$ is bounded and every hyperbolic triangle has area no more than π. Moreover, it is easy to check that $\phi_{p,\alpha}$ only changes by a uniformly bounded amount if we change the base point, thus ϕ_α is a quasimorphism and is the homogenization of $\phi_{p,\alpha}$. This gives an injection from the (huge) space of smooth 1-forms to $Q(\pi_1(M))$.

The following generalized Bavard's duality theorem links scl and quasimorphisms:

Theorem 4.5 (Bavard's duality, [4])**.** *For any $c = \sum t_i g_i \in B_1^H(G)$, we have*

$$\mathrm{scl}(c) = \sup_{\phi\in Q(G)/H^1(G)} \frac{\sum t_i\phi(g_i)}{2D(\phi)}.$$

Remark 4.6. (a) We see in particular that scl vanishes on G iff $Q(G) = H^1(G)$. More generally, when $Q(G)/H^1(G)$ is finite dimensional, calculation of scl is just maximizing finitely many linear functions, and scl is piecewise linear in this case. We will see when this happens in the following subsection.
(b) Compare this to Proposition 3.2. An efficient admissible surface would give a good upper bound on scl, while a suitable quasimorphism would give a nice lower bound. Although extremal surface may not exist, extremal quasimorphism always exists.

By the duality theorem, it is good to have some tools to determine what $Q(G)$ is and construct some interesting quasimorphisms. These are what we do in the following two subsections respectively.

4.1. **Bounded Cohomology.** In this subsection we briefly introduce bounded cohomology and its relation to quasimorphisms, and we will see that amenable groups have no nontrivial homogeneous quasimorphisms and thus scl vanishes on them.

Bounded cohomology is introduced by Gromov ([16]) as the dual to simplicial norm (also called Gromov's norm).

Definition 4.7. Recall from group homology that $C_n(G;\mathbb{R})$ is the $\mathbb{R}$-vector space with basis G^n (as a set), and there is a linear map $\partial : C_n(G;\mathbb{R}) \to C_{n-1}(G;\mathbb{R})$ determined by

$$\partial(g_1,\dots,g_n) = (g_2,\dots,g_n)+\sum_{i=1}^{n-1}(-1)^i(g_1,\dots,g_{i-1},g_ig_{i+1},g_{i+2},\dots,g_n)+(-1)^n(g_1,\dots,g_{n-1}).$$

Then $\partial^2 = 0$ and $(C_*(G;\mathbb{R}),\partial)$ is a complex whose homology is the group homology of G.

Note that $C_n(G;\mathbb{R})$ is a vector space with canonical basis, we can equip it with the L^1 norm, then it induces a pseudo-norm on $H_n(G;\mathbb{R})$, called the simplicial norm.

A similar definition works for the real simplicial homology of topological spaces.

Example 4.8. Let M be an oriented closed manifold, then the simplicial norm of the fundamental class $[M]$ is called the Gromov's volume of M. If M is S^n or T^n, then the Gromov's volume is 0 since M admits self maps with degree larger than 1. If M is a n-dimensional closed hyperbolic manifold, then Gromov's proportionality theorem states that the Gromov's volume is a constant v_n multiple of the hyperbolic volume of M (does not depends on the choice of hyperbolic structure!), and $v_n = \sup \mathrm{vol}(\Delta)$ over all hyperbolic n-simplices, which turns out ([17]) to be the volume of the regular ideal n-simplex. This implies that the hyperbolic volume of a closed hyperbolic manifold is a homotopy invariant, which also can be seen from Mostow rigidity.

In particular, we see the Gromov's volume of a closed oriented surface S is $-2\pi\chi^-(S)$. Using this, we can easily get the following interesting corollary, which does not seem immediate from classical algebraic topology.

Corollary 4.9. Let S and S' be closed oriented surfaces, then

$$\{d\in\mathbb{Z} | \exists f : S \to S' \text{ s.t. } \deg(f) = d\} = \{d \in \mathbb{Z} | -\chi^-(S)|d| \geq -\chi^-(S')\}.$$

Note that there are non-compact manifolds of finite volume, but the Gromov's volume defined above does not quite make sense here. One waay to deal with this is to take the L^1 completion of C_n and work with L^1-homology. Instead of this, Gromov ([16]) suggests to work with its dual, bounded cohomology.

Definition 4.10. The dual complex $(C^*(G;\mathbb{R}),\delta)$ of $C_*(G;\mathbb{R})$ gives the group cohomology. Inside $C^n(G;\mathbb{R})$, we have the subspace $C^n_b(G;\mathbb{R})$ consisting of bounded cochains, and it is Banach with the sup norm $\|\cdot\|_\infty$. Moreover, δ restricts to a bounded linear map $\delta : C^n(G;\mathbb{R}) \to C^{n+1}(G;\mathbb{R})$, and the cohomology of the complex $(C^*_b(G;\mathbb{R}),\delta)$ are denoted by $H^n_b(G;\mathbb{R})$, called *the bounded cohomology.* $\|\cdot\|_\infty$ induces a pseudo-norm on H^*_b.

Note that $C^1_b(G;\mathbb{R})$ is the space of bounded maps to $\mathbb{R}$, $H^1(G;\mathbb{R})$ is the space of homomorphisms to $\mathbb{R}$, both of them are subspace of $\widehat{Q}(G) \subset C^1(G;\mathbb{R})$ and their intersection is $H^1_b(G;\mathbb{R}) = 0$. Also, homogenization shows $\widehat{Q}(G) \cong C^1_b(G;\mathbb{R}) \oplus Q(G)$.

Also note that $\delta\phi(a,b) = \phi(a)+\phi(b)-\phi(ab)$ for $\phi \in C^1$ and $a,b\in G$, thus $\|\delta(\phi)\|_\infty = D(\phi)$ by definition and $\delta(\phi) \in C^2_b(G;\mathbb{R})$ if and only if ϕ is a quasimorphism. From this it is easy to see:

Proposition 4.11. *We have an exact sequence:*

$$0 \to H^1(G;\mathbb{R}) \to Q(G) \to H^2_b(G;\mathbb{R}) \to H^2(G;\mathbb{R}),$$

where the last map is induced by inclusion at the level of cochains.

Thus from Bavard's duality, $H^2_b(G;\mathbb{R}) \to H^2(G;\mathbb{R})$ is injective if and only if scl vanishes. It is proved by Burger and Monod that many lattices in higher rank non-compact real Lie groups satisfies this condition.

Bounded cohomology behaves nicely under taking amenable cover:

Theorem 4.12 (Johnson, Trauber, Gromov, [16][18])**.** *For any short exact sequence of groups*

$$1 \to H \to G \to A \to 1$$

where A is amenable, the natural homomorphisms $H^*_b(G;\mathbb{R}) \to H^*_b(H;\mathbb{R})^A$ are isometric isomorphisms in each dimension, where the action of A on $H^*_b(H;\mathbb{R})$ comes from the action of A on H by outer automorphisms.

In particular, by taking $H = 1$ and $G = A$ in the theorem, we get

Corollary 4.13. $H^*_b(A;\mathbb{R}) = 0$ for all amenable group A, thus scl_A vanishes. Moreover, if A is also discrete, then scl_K vanishes for any subgroup K of A since it is also amenable.

Note that there are interesting stories between "amenability" and "no free non-abelian subgroups", we have the following implications for any discrete group G:

G is amenable $\implies$ scl_H vanishes $\forall H \leq G$ $\implies$ G has no free non-abelian subgroup

The converse of the first implication has counter examples among finitely generated groups. It is shown by Adian ([1]) that the Burnside's group $B(m,n)$ is not amenable for $m \geq 2$ and $n \geq 665$, but scl_H vanishes for any subgroup since $B(m,n)$ is a torsion group. The converse of the second implication is an interesting open problem. It is equivalent to the following:

Question 4.14. Let G be a (say, discrete) group that has no free non-abelian subgroup, does scl_G vanish?

There is another exact sequence for second bounded cohomology:

Theorem 4.15 (Bouarich, [5])**.** *Any exact sequence*

$$K \to G \to H \to 1$$

induces an exact sequence of second bounded cohomology

$$1 \to H^2_b(H;\mathbb{R}) \to H^2_b(G;\mathbb{R}) \to H^2_b(K;\mathbb{R}).$$

In particular, if K is amenable, we have an isomorphism $H^2_b(H;\mathbb{R}) \cong H^2_b(G;\mathbb{R})$.

Remark 4.16. From this we see, if $1 \to A \to \widehat{G} \to G \to 1$ is an central extension, then $Q(\widehat{G})/H^1(\widehat{G};\mathbb{R})$ sits inside $H^2_b(\widehat{G};\mathbb{R}) \cong H^2_b(G;\mathbb{R})$. In particular, if G is finitely presented and scl_G vanishes, then $H^2_b(G;\mathbb{R})$ is finite dimensional since it injects into $H^2(G;\mathbb{R})$, therefore $Q(\widehat{G})/H^1(\widehat{G};\mathbb{R})$ is also finite dimensional and scl on $\widehat{G}$ can be calculated by Bavard's duality.

4.2. **Examples and Constructions of Quasimorphisms.** Now we give some interesting examples of quasimorphisms from different sources, such as dynamics and geometry of Gromov hyperbolic spaces.

4.2.1. *From Dynamics.* Here we typically consider the group of certain transformations on some space.

First we take $G = \text{Homeo}^+(S^1)$. Suppose fix_g, the set of fixed points under $g \in G$, is nonempty, then $S^1 - \text{fix}_g$ is a union of open intervals, and g restricted to each of them is topologically conjugate to a translation on $\mathbb{R}$, which is a commutator of two dilations. Thus g can be written as a commutator in G. Now if g has no fixed point, take any $p \in S^1$, we may find a commutator in G that maps p to $g(p)$. Thus any $g \in G$ has commutator length no more than 2, so we get

Proposition 4.17. *$G = Homeo^+(S^1)$ is uniformly perfect, thus $H^1(G;\mathbb{R}) = 0$,* scl *vanishes on G, and $Q(G) = 0$ by Bavard's duality.*

Now consider $\widehat{G}$, the subgroup of $\text{Homeo}^+(\mathbb{R})$ that descends to maps in G under the usual covering map $\mathbb{R} \to S^1$ by mod $\mathbb{Z}$. Then the translation $x \mapsto 1$ generates a subgroup isomorphic to $\mathbb{Z}$ lying in the center of $\widehat{G}$, and we get a central extension

$$1 \to \mathbb{Z} \to \widehat{G} \to G \to 1.$$

Apply the proposition above together with the two exact sequences from Proposition 4.11 and Theorem 4.15, and note that the Euler class of such a nontrivial extension gives an element in the kernel of $H^2(G;\mathbb{R}) \to H^2(\widehat{G};\mathbb{R})$, we get

Proposition 4.18. *$Q(\widehat{G}) \cong \mathbb{R}$ and $H^1(\widehat{G};\mathbb{R}) = 0$.*

Now we explicitly describe a generator of $Q(\widehat{G})$. For any $p \in \mathbb{R}$, define $t_p(\phi) = \phi(p) - p$ for all $\phi \in \widehat{G}$. Then it is easy to check t_p is a quasimorphism and $t_p - t_q$ is uniformly bounded, thus these quasimorphisms have the same homogenization $\text{rot} : \widehat{G} \to \mathbb{R}$, called the rotation number given by $\text{rot}(\phi) = \lim \phi^n(0)/n$.

Proposition 4.19. *$Q(\widehat{G}) \cong \mathbb{R}$ is generated by* rot*, the unique homogeneous quasimorphism on $\widehat{G}$ taking the unit translation to* 1*.*

Remark 4.20. rot descends to a map $G \to \mathbb{R}/\mathbb{Z}$ also called rotation number, which is originally defined by Poincaré, who showed that the dynamical property of any $\phi \in G$ is completely determined by the rationality of the rotation number of ϕ.

A careful analysis shows that $\sup_{a,b} |t_p([a,b])| = 2$. This on the one hand implies $|t_x(g^n)| \leq 2\text{cl}(g^n)$ and thus $|\text{rot}| \leq 2\text{scl}$. On the other hand, it also implies that we can multiply at most $[|\text{rot}(\phi)|/2 + 1]$ commutators to ϕ to make it fix 0 and then it becomes a commutator by the trick we used at the beginning of this subsubsection, hence $2\text{scl} \leq |\text{rot}|$. Thus Bavard's duality implies

Proposition 4.21. *$D(\text{rot}) = 1$ on $\widehat{G}$. If we pull back* rot *by any homomorphism $f : K \to \widehat{G}$, then $D(f^*\text{rot}) \leq 1$ as a quasimorphism on K.*

In particular, let S be a compact oriented hyperbolic surface with geodesic boundary, we get a homomorphism $\rho : \pi_1(S) \to \text{Isom}^+(\mathbb{H}^2) \leq G$, then rot pulls back to a quasimorphism rot_ρ on $\pi_1(S)$. It turns out that rot_ρ takes discrete value and thus does not depend on the choice of hyperbolic metric. Moreover, it can be interpreted as a sort of algebraic intersection number. Therefore we denote this quasimorphism simply by rot_S. Amazingly, whether a loop virtually bounds an immersed surface in S is exactly detected by this quasimorphism.

Theorem 4.22 (Calegari, [6])**.** *A rational chain in $C \in B_1^H(\pi_1(S))$ represented by a weighted sum of geodesics Γ virtually bounds a (positive or negative) immersed surface in S*

if and only if rot_S *is extremal for* C*, i.e.* $\mathrm{scl}(C) = |\mathrm{rot}_S(C)|/2$*. Moreover, for all* N *sufficiently large (depending on* C*),* $N\partial S + \Gamma$ *virtually bounds a (positive) immersed surface. Finally, if* S *has no boundary, any such* Γ *virtually bounds a (positive) immersed surface.*

Such a construction can be generalized to a symplectic rotation number. Consider the standard symplectic form $\omega = \sum x^i \wedge y^i$ on $\mathbb{R}^{2n}$. Recall that a n-dimensional subspace V is Lagrangian if and only if ω restricts to 0. Let Λ_n denote the space of Lagrangians, then the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$ acts on it. Identifying $\mathbb{R}^{2n}$ with $\mathbb{C}^n$, V is Lagrangian if and only if it is totally real, i.e. V and iV are orthogonal complements in $\mathbb{R}^{2n}$, or in other words, a real orthonormal basis on V is a complex orthonormal basis of $\mathbb{C}^n$. From this we immediately see that $\mathrm{U}(n)$ acts on Λ_n, the space of Lagrangians, transitively with point stabilizer $\mathrm{O}(n)$. Hence we can identify Λ_n with $\mathrm{U}(n)/\mathrm{O}(n)$.

Now take any base point $b \in \Lambda_n$, e.g. $\mathbb{R}^n \times \{0\}$. Notice that $\det^2 : \mathrm{U}(n) \to S^1$ induces a map $\det^2 : \Lambda_n \to S^1$, which gives a map $\mu : \mathrm{Sp}(2n, \mathbb{R}) \to S^1$ by $\mu(g) = \det^2(g(b))$. Note that the inclusion $\mathrm{U}(1) \hookrightarrow \mathrm{Sp}(2n, \mathbb{R})$ induces an isomorphism on π_1, and μ restricted to $\mathrm{U}(1)$ is the double cover. μ lifts to $\tilde{\mu} : \widetilde{\mathrm{Sp}}(2n) \to \mathbb{R}$ which turns out to be a quasimorphism ([3]) with homogenization (after normalizing, it is called the symplectic rotation number) that spans $Q(\widetilde{\mathrm{Sp}}(2n))$.

4.2.2. *From Hyperbolic Geometry.* We have already seen from Example 4.4 that the fundamental group of a closed hyperbolic manifold has lots of quasimorphisms. Moreover, one can use these quasimorphisms to show that scl is a genuine norm (instead of a pseudo-norm) on such groups. Thus it is natural to guess that word hyperbolic groups (and more generally, groups acting nicely on Gromov hyperbolic spaces) might also have lots of interesting quasimorphisms.

Consider a group G acting simplicially on a δ-hyperbolic (not necessarily locally finite) complex X with simplicial metric d.

Definition 4.23. For any finite simplicial path γ, let $\ell(\gamma)$ be the simplicial length of γ, which is an integer. For any finite oriented simplicial path, a *copy* of σ is $a \cdot \sigma$ for some $a \in G$, and σ^{-1} is σ with opposite orientation. Let $|\gamma|_\sigma$ be the maximal number of disjoint copies of σ in γ. For any $p, q \in X$, define

$$c_\sigma([p,q]) = d(p,q) - \inf_\gamma (\ell(\gamma) - |\gamma|_\sigma),$$

where the infimum is taken over all simplicial paths from p to q.

Remark 4.24. (a) $c_\sigma([p,q])$ roughly measures the maximal number of disjoint copies of σ one can have on "the" geodesic from p to q. But there might be no unique geodesic joining two points, thus all paths are taken into consideration.
(b) Since ℓ and $|\cdot|_\gamma$ take integer values, the infimum must be achieved by some paths, referred to as *realizing paths*. We will see below that realizing paths are quasi-geodesics when $\ell(\sigma) \geq 2$.

Example 4.25. Let G be a hyperbolic group and X be its Cayley graph with respect to some generating set. Then X is a δ-hyperbolic complex with G acting simplicially and properly discontinuously. In particular, take $G = F_2$ be the free group generated by a, b and X be the Cayley graph for generating set $\{a, b, a^{-1}, b^{-1}\}$. A finite oriented simplicial path σ and its copies correspond to a reduced word $w \in F_2$, and since X is a tree, it is immediate to check that $c_\sigma([p,q])$ is just the maximal number of disjoint word w in qp^{-1}.

Lemma 4.26. *The following estimates can be easily checked from the definition*

(1) $c_\sigma([p,q]) = c_{\sigma^{-1}}([q,p])$;

(2) $|c_\sigma([p,q]) - c_\sigma([p,q'])| \leq d(q,q')$;

(3) *If q is on a realizing path for σ from p to r, then*
$$c_\sigma([p,r]) \geq c_\sigma([p,q]) + c_\sigma([q,r]) \geq c_\sigma([p,r]) - 1.$$

When $\ell(\sigma) \geq 2$, we have $\ell(\gamma) - |\gamma|_\sigma \geq \ell(\gamma)/2$, from this and the estimates above, one can show that

Lemma 4.27 (Fujiwara, [15])**.** *If $\ell(\sigma) \geq 2$, then any realizing path for c_σ is a $(2,4)$-quasi-geodesic. By Morse lemma, there is a constant $C(\delta)$, such that if the $C(\delta)$ neighborhood of any geodesic from p to q does not contain a copy of σ, then $c_\sigma([p,q]) = 0$.*

Definition 4.28. Fix a base point $b \in X$, define $c_\sigma(g) = c_\sigma([b, gb])$ and $h_\sigma(g) = c_\sigma(g) - c_{\sigma^{-1}}(g)$ for any $g \in G$.

Example 4.29. Continue the example above and take the identity to be the base point, then $c_\sigma(g)$ is just the maximal number of the reduced word w corresponding to σ one can have in the reduced word g. Then it is hands on to show h_σ is a quasimorphism on F_2 called the little counting quasimorphism, and h_σ is the generalization of such quasimorphisms. Careful analysis shows that the defect of a little counting quasimorphism is either 1 or 2 when w has length at least 2.

As one can expect, h_σ is a quasimorphism and the defect has an uniform upper bound.

Lemma 4.30 (Fujiwara, [15])**.** *If $\ell(\sigma) \geq 2$, then there is a constant $C(\delta)$ such that $D(h_\sigma) \leq C$.*

However, it is still not clear up to this point whether we get some nontrivial quasimorphisms from such a construction. It turns out that when an element g acts in a "nice" way on X and has no power conjugate to its inverse, then we can get a nontrivial quasimorphism. Here the word "nice" refers to the following notion:

Definition 4.31. With G and X as above, we say the action of $g \in G$ is *weakly properly discontinuously* if for any $x \in X$ and $C > 0$, there is a constant N such that the set
$$\{f \in G | d(x, fx) \leq C \text{ and } d(g^N x, f g^N x) \leq C\}$$
is finite.

From this, we get lots of quasimorphisms on: hyperbolic groups by acting on the Cayley graph, mapping class groups of closed oriented hyperbolic surfaces by acting on the complex of curves, and the group of orientation preserving homeomorphisms on the complement of a Cantor set in $\mathbb{R}^2$ by acting on the ray graph.

On the other hand, it is necessary to have weakly properly discontinuous action: $\mathrm{SL}(2,\mathbb{Z}[1/2])$ admits a simplicial and minimal action on a tree since it is an amalgam, but it is shown that there is no nonzero homogeneous quasimorphism.

Using such quasimorphisms, it is shown that

Theorem 4.32 (Calegari-Fujiwara, [11])**.** *If G is δ-hyperbolic with respect to generating set A, then* scl *is a norm on $B_1^H(G)$. Moreover, there is a spectral gap $C = C(\delta, |A|) > 0$:* $\mathrm{scl}(c) \geq C$ *for all nonzero integral chain $c \in B_1^H(G)$.*

References

1. Adian S.I, *The Burnside Problem and Identities in Groups*, Berlin; New York: Springer-Verlag, 1979.
2. Jean Barge and Étienne Ghys, *Surfaces et cohomologie bornée*, Invent. Math. **92** (1988), no. 3, 509–526.
3. Jean Barge and Étienne Ghys, *Cocycles d'Euler et de Maslov*,Math. Ann. **294** (1992), no. 2, 235–265.
4. C. Bavard, *Longeur stable des commutateurs*, LEnseign. Math. **37** (1991), 109–150.
5. Abdessalam Bouarich, *Suites exactes en cohomologie bornée réelle des groupes discrets*, C. R. Acad. Sci. Paris Sér. I Math. **320** (1995), no. 11, 1355–1359.
6. D. Calegari, *Faces of the scl norm ball*, Geom. Topol. **13** (2009), 1313–1336.
7. D. Calegari, *Stable commutator length is rational in free groups*, Jour. AMS **22** (2009), no. 4, 941–961.
8. D. Calegari, *Scl, sails and surgery*, Jour. Topology **4** (2011), no. 2, 305–326.
9. D. Calegari, *scl*, MSJ Memoirs, **20**. Mathematical Society of Japan, Tokyo, 2009.
10. D. Calegari, *Stable commutator length in subgroups of* $\mathrm{PL}^+(I)$, Pacific J. Math. **232** (2007), no.3, 257–262.
11. D. Calegari and K. Fujiwara, *Stable commutator length in word hyperbolic groups*, Groups, Geom., Dynam. **4** (2010) no. 1, 59–90.
12. D. Calegari and A. Walker, *Isometric endomorphisms of free groups*, New York J. Math. **17** (2011), 713–743.
13. D. Calegari and A. Walker, `scallop`, computer program available from the authors webpages, and from computop.org.
14. Andrew J. Duncan and James Howie, *The genus problem for one-relator products of locally indicable groups*, Math. Z. **208** (1991), no. 2, 225–237.
15. Koji Fujiwara, *The second cohomology of a group acting on a Gromov-hyperbolic space*, Proc. London Math. Soc. (3) **76** (1998), no. 1, 70–94.
16. Mikhael Gromov, *Volume and bounded cohomology*, Inst. Hautes Études Sci. Publ. Math. (1982), no. 56, 5–99 (1983).
17. U. Haagerup and H. Munkholm, *Simplices of maximal volume in hyperbolic* n*-space*, Acta Math. **147** (1981), no. 1–2, 1–11.
18. Barry Edward Johnson, *Cohomology in Banach algebras*, American Mathematical Society, Providence, R.I., 1972, Memoirs of the American Mathematical Society, No. 127.
19. Martin Liebeck, Eamonn O'Brian, Aner Shalev, and Pham Tiep, *The Ore conjecture*, J. Eur. Math. Soc. **12** (2010), no. 4, 939–1008.
20. T. Susse, *Stable commutator length in amalgamated free products*, J. Topol. Anal. **7** (2015), no. 04, 693–717.
21. A. Walker, *Stabel commutator length in free products of cyclic groups*, Experimental Math **22** (2013), no. 3, 282–298.
22. A. Walker, `wallop`, computer program available from the authors webpages, and from computop.org.
23. D. Zhuang, *Irrational stable commutator length in finitely presented groups*, J. Mod. Dyn. **2** (2008), no. 3, 497–505.

Department of Mathematics, University of Chicago, Chicago, Illinois, USA, 60637
E-mail address: `lzchen@math.uchicago.edu`