

# NIELSEN REALIZATION FOR INFINITE-TYPE SURFACES

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ABSTRACT. Given a finite subgroup  $G$  of the mapping class group of a surface  $S$ , the Nielsen realization problem asks whether  $G$  can be realized as a finite group of homeomorphisms of  $S$ . In 1983, Kerckhoff showed that for  $S$  a finite-type surface, any finite subgroup  $G$  may be realized as a group of isometries of some hyperbolic metric on  $S$ . We extend Kerckhoff's result to orientable, infinite-type surfaces. As applications, we classify torsion elements in the mapping class group of the plane minus a Cantor set, and also show that topological groups containing sequences of torsion elements limiting to the identity do not embed continuously into the mapping class group of  $S$ . Finally, we show that compact subgroups of the mapping class group of  $S$  are finite, and locally compact subgroups are discrete.

In 1932, Nielsen asked whether finite subgroups of mapping class groups act on surfaces. In 1983, Kerckhoff [Ker83] gave the following strong affirmative answer.

**Theorem 1** (Kerckhoff, Theorem 5 [Ker83]). *Let  $S$  be a finite-type surface with negative Euler characteristic. Any finite subgroup of the mapping class group of  $S$  may be realized as a group of isometries of some hyperbolic metric on  $S$ .*

Let  $S$  be a surface. We distinguish two kinds of surfaces, saying  $S$  is a *finite-type* surface if its fundamental group is finitely generated, and is an *infinite-type* surface otherwise. Recently there has been a surge of interest in infinite-type surfaces and their mapping class groups. We refer the interested reader to a recent survey by Aramayona and Vlamis [AV20].

The main purpose of this paper is to extend Kerckhoff's result to the infinite-type case.

**Theorem 2.** *Let  $S$  be an orientable, infinite-type surface whose boundary is either empty or a union of circles. Any finite subgroup of the mapping class group of  $S$  may be realized as a group of isometries of some hyperbolic metric on  $S$ .*

Let us briefly sketch the idea of the proof. Let  $G$  be a finite subgroup of the mapping class group of  $S$ . The mapping class group acts on the *Teichmüller space* of  $S$ , denoted  $\mathcal{T}(S)$ , which parameterizes hyperbolic structures on  $S$  up to isotopy. Kerckhoff defines a  $G$ -invariant map  $\ell_G: \mathcal{T}(S) \rightarrow \mathbb{R}_+$ . The map sends a hyperbolic structure to the sum of the geodesic lengths of a certain finite  $G$ -invariant collection of simple closed curves on  $S$ . When  $S$  is of finite type, Kerckhoff proves that the map  $\ell_G$  attains a unique minimum. This minimum is fixed by  $G$ , yielding an action of  $G$  by isometries of the corresponding hyperbolic structure on  $S$ .

In the infinite-type setting, all of the tools in Kerckhoff's proof are available to us, but the sum defining  $\ell_G$  diverges. Instead, we find an exhaustion of  $S$  by connected,  $G$ -invariant (homotopy classes of) finite-type subsurfaces  $S_0 \subset S_1 \subset \dots$ . Kerckhoff's theorem applies to each piece  $\overline{S_k} \setminus \overline{S_{k-1}}$ , and we show how to assemble the pieces to give a hyperbolic structure on  $S$  and an action of  $G$  by isometries. It would be interesting to know if Kerckhoff's method of proof could be applied more directly.

We remark that if  $\Sigma$  is a finite-type subsurface of  $S$ , the hyperbolic metric on  $S$  in the theorem is chosen so that  $\Sigma$  inherits a hyperbolic metric of finite volume. In other words, each isolated end of  $S$  is given a metric modeled on the pseudosphere rather than a flared annulus.

Let us mention recent work of Aougab, Patel and Vlamis. In [APV20], they study a similar realization problem: given a group  $G$  and a surface  $S$ , is there a hyperbolic metric on  $S$  whose isometry group is isomorphic to  $G$ ? They show that for many infinite-type surfaces, every countable group  $G$  can in fact be realized in this way. Of course, note that their result does not preclude the existence of other embeddings of  $G$  into  $\text{Map}(S)$  which cannot be realized as groups of isometries.

**Corollary 3.** *If  $S$  is an orientable, infinite-type surface with nonempty compact boundary, the relative mapping class group fixing the boundary pointwise is torsion-free.*

As an application, the Nielsen realization theorem allows us to classify torsion elements in the mapping class group of the plane minus a Cantor set; see Theorem 7.

Equip the full homeomorphism group of  $S$  with the compact-open topology, and the mapping class group of  $S$  with the quotient topology. As another application, we have the following two theorems.

**Theorem 4.** *If  $G$  is a topological group containing a sequence of nontrivial finite order elements limiting to the identity, then  $G$  does not embed (as a topological group) in the mapping class group of  $S$ .*

**Theorem 5.** *Compact subgroups of  $\text{Map}(S)$  are finite, and locally compact subgroups are discrete.*

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In the remainder of this introduction, let us make the statement of Theorem 2 more precise. The *mapping class group* of  $S$ , denoted  $\text{Map}(S)$ , is the group  $\pi_0(\text{Homeo}(S))$  of homotopy classes of homeomorphisms  $f: S \rightarrow S$ . If  $S$  has nonempty boundary  $\partial S$ , let  $\text{Map}(S, \partial S)$  denote the group of homotopy classes of homeomorphisms  $f: S \rightarrow S$  where we require all homeomorphisms and homotopies to fix the boundary  $\partial S$  pointwise.

A classification of infinite-type surfaces was given by Kerékjártó [Ker23] and Richards [Ric63]. We recall that each surface  $S$  has a *space of ends*, defined as usual as an inverse limit  $\varprojlim \pi_0(S \setminus K)$  as  $K$  ranges over the compact subsets of  $S$ . If we fix a compact exhaustion  $K_0 \subset K_1 \subset \dots$  of  $S$ , an *end* is represented as a sequence

$$U_0 \supset U_1 \supset \dots,$$

where each  $U_i$  is a connected component of  $S \setminus K_i$ . The end is *isolated* if all but finitely many of the  $U_i$  have one end, and *planar* if all but finitely many of the  $U_i$  are planar. These properties do not depend on the choice of compact exhaustion. Finite-type surfaces have finitely many isolated planar ends.

By a *hyperbolic structure* on a surface  $S$ , we mean  $S$  equipped with a complete Riemannian metric of constant curvature  $-1$  satisfying the following two conditions. (i) We require that each isolated planar end of  $S$  is modeled on the pseudosphere, and following Kerckhoff [Ker83, Section 4], (ii) we require that each boundary curve is a geodesic with length 1. In the case where  $S$  is of finite type, the condition on ends is satisfied by insisting that the

metric has finite volume. We remark that the condition that boundary curves have length 1 is slightly nonstandard but useful. In addition, forcing this condition on our finite-type exhaustion of  $S$  places the hyperbolic structure on  $S$  in the component containing the “thick part” of Teichmüller space.

### 1. FINDING AN INVARIANT EXHAUSTION

Fix  $S$  an orientable, connected, infinite-type surface whose boundary is empty or a union of circles and fix  $G$  a nontrivial finite subgroup of  $\text{Map}(S)$ . The goal of this section is to prove the following proposition.

**Proposition 6.** *There exists an exhaustion of  $S$  by connected, finite-type subsurfaces  $\emptyset = S_0 \subset S_1 \subset \dots$  such that for each  $k \geq 1$ ,  $S_k$  is  $G$ -invariant up to homotopy, and each component of  $S_k \setminus S_{k-1}$  has negative Euler characteristic.*

This proposition is crucial to our proof of Theorem 2.

*Proof.* First, fix an arbitrary exhaustion  $\{K_i\}$  of  $S$  by connected, finite-type subsurfaces, e.g. coming from a pants decomposition of  $S$ . Let  $X$  be a subset of  $\text{Diff}(S)$  containing a single representative from each mapping class in  $G$ . We will proceed by induction on the length of a chain of subsurfaces  $S_0 \subset S_1 \subset \dots \subset S_n$ . For the base case, set  $S_0 = \emptyset$ .

Since  $S$  is of infinite type, there exists some essential, simple closed curve  $\alpha$  on  $S \setminus S_n$ . Recall that a simple closed curve is *essential* if it does not bound a disk nor is homotopic to a boundary curve or a puncture. Moreover, there exists some term  $K_{i_n}$  of the exhaustion containing both the  $X$ -orbit of  $\alpha$  and  $S_n$ , since  $X$  is finite. To the end of creating a  $G$ -invariant exhaustion, consider

$$K = \bigcap_{f \in X} f(K_{i_n}).$$

Note that  $K$  contains the  $X$ -orbit of  $\alpha$  as well as  $S_n$ . By performing a preliminary isotopy, we may assume the  $f(K_{i_n})$  pairwise intersect transversely, and thus  $K$  is itself a subsurface. Since  $f(K) = K$  for each  $f \in X$  by construction, we may take  $S_{n+1}$  to be  $K$ , together with the finitely many components of  $S \setminus K$  that contain no essential, simple closed curve. If any component of  $S_{n+1} \setminus S_n$  is an annulus, by induction we may assume that the component of  $S \setminus S_n$  it determines contains an essential simple closed curve. Thus there is no loss in replacing  $S_{n+1}$  by an isotopic subsurface so that no component of  $S_{n+1} \setminus S_n$  is an annulus.

Choosing the essential simple loop  $\alpha$  appropriately at each step, the inductive construction above gives an exhaustion of  $S$ . □

### 2. NIELSEN REALIZATION

We continue with the notation from the previous section. Choose a hyperbolic structure on  $S$  so that each  $S_i$  has finite volume and each component of  $\partial S_i$  is a geodesic with length 1. Write  $P_i = \overline{S_i \setminus S_{i-1}}$ . Since each  $S_i$  is  $G$ -invariant up to isotopy, so is each  $P_i$ . Each  $g \in G$  gives a well-defined mapping class in  $\text{Map}(P_i)$  by choosing an arbitrary homotopy between  $P_i$  and  $x_g(P_i)$ , where  $x_g$  is a diffeomorphism on  $S$  representing  $g$ . Thus each  $g \in G$  defines a mapping class  $\rho(g)$  in  $\text{Map}(P_i)$ . The inclusion  $G \rightarrow \text{Map}(P_i)$  is actually injective and follows from the proof below (see e.g. Proposition 9). Our proof does not depend on this fact.

*Proof of Theorem 2.* According to Kerckhoff, there is another hyperbolic structure on each  $P_i$  with respect to which the image of  $G$  in  $\text{Map}(P_i)$  may be realized as a group of isometries [Ker83, Theorem 5, discussion following Theorem 4]. Note that each boundary curve of each  $P_i$  remains geodesic with length 1.

Let us say a few words about the case where  $P_i$  is disconnected. Let  $C$  be a component of  $P_i$ , and write  $H$  for the stabilizer of  $C$  in  $G$ . By the above, there is a hyperbolic structure on  $C$  with an action of  $H$  by isometries. Choose a set of coset representatives for  $G/H$ . If  $C'$  is a component of the orbit of  $C$  distinct from  $C$  choose a representative in  $X$  taking  $C'$  to  $C$ , and give  $C'$  a new hyperbolic structure by pulling back the metric from  $C$ . Proceeding orbit by orbit, this yields a realization of  $G$  on  $P_i$ .

We now have a hyperbolic structure for each  $P_i$  with an action of  $G$  by isometries. Gluing up the  $P_i$  via the identifications coming from  $S$  yields a hyperbolic structure on  $S$ , but we need to do so respecting the action of  $G$ . As above, we glue the boundary curves shared by  $P_i$  and  $S_{i-1}$  orbit by orbit. It suffices to show that for each boundary curve  $c$  in  $P_i \cap S_{i-1}$  there is a gluing that identifies the two circle actions on  $c$  by its stabilizer  $\text{Stab}(c)$ . Note that each  $g \in \text{Stab}(c)$  acts on  $c$  by rigid rotations, and with the (opposite) orientations induced from  $P_i$  and  $S_{i-1}$ , it suffices to show that the angles of rotations for the two actions differ by a negative sign. Indeed, on the  $P_i$  (resp.  $S_{i-1}$ ) side, the angle is the rotation number of the  $g$  action on the circle of geodesic rays on  $P_i$  (resp.  $S_i$ ) starting from and perpendicular to  $c$ . This action is similar to the one on the conical circle (see e.g. [BW18b]), and only depends on the mapping class  $g$ . Thus the two angles differ by a negative sign due to the opposite orientations.

As a result, we have an isometric action of  $G$  on each  $P_i$  that respects the gluing, yielding a hyperbolic structure and an isometric action of  $G$  on  $S$ .  $\square$

### 3. CLASSIFICATION OF TORSION ELEMENTS

If  $S$  is a surface of finite genus and empty boundary, then by the classification theorem [Ric63] it is realized as  $\Sigma - E$ , where  $\Sigma$  is the closed surface with the same genus as  $S$  and  $E$  is a totally disconnected closed subset of  $\Sigma$  homeomorphic to the space of ends. In this case, Theorem 2 implies that any finite subgroup  $G$  of  $\text{Map}(S)$  is realized by some  $G$ -action on  $\Sigma$  by homeomorphisms preserving  $E$ . This is because  $\text{Homeo}^+(\Sigma - E) \cong \text{Homeo}^+(\Sigma, E)$ , where the latter denotes orientation-preserving homeomorphisms of  $\Sigma$  preserving  $E$ .

In particular, one can use Theorem 2 to classify torsion elements. Here we focus on an example, the case of  $S = \mathbb{R}^2 - K$ , where  $K$  is a Cantor set. In this situation, the mapping class group acts faithfully on the *conical circle*  $S_C^1$  consisting of geodesics (for a fixed complete hyperbolic metric on  $S$ ) emanating from  $\infty$ ; see e.g. [BW18a, CC20]. Thus each mapping class  $g$  has a rotation number, which can be read off from its action on a special subset of  $S_C^1$ , namely the *short rays*, which are proper simple geodesics connecting  $\infty$  to some point in the Cantor set.

**Theorem 7.** *Let  $S = \mathbb{R}^2 - K$ , where  $K$  is a Cantor set. Finite order elements of  $\text{Map}(S)$  fix at most one point in  $K$ . For each  $n \geq 2$ , elements in  $\text{Map}(S)$  of order  $n$  fall into  $2\varphi(n)$  conjugacy classes, which are distinguished by the rotation number and whether the element fixes exactly one point in  $K$  or none. Here  $\varphi(n)$  is the number of positive integers up to  $n$  that are coprime to  $n$ .*

*Proof.* Let  $g \in \text{Map}(S)$  be an element of order  $n$ . Then the action of  $g$  on the conical circle  $S_C^1$  has rotation number  $m/n \pmod{\mathbb{Z}}$  for some  $m$  coprime to  $n$ . By Theorem 2, we can

realize  $g$  as some  $\tilde{g} \in \text{Homeo}^+(S^2, K \cup \{\infty\})$  of order  $n$ . It is known that any finite order homeomorphism on  $S^2$  is conjugate to a rigid rotation [Zim12] and the quotient  $S^2/\tilde{g}$  is still homeomorphic to  $S^2$ . Considering the rotation number, we conclude that  $\tilde{g}$  is conjugate to a rigid rotation by  $2m\pi/n$ , and there is exactly one fixed point  $p \in S^2$  other than  $\infty$ .

We can put a Cantor set on  $S^2$  invariant under a rigid rotation on  $S^2$  by an angle of  $2m\pi/n$  fixing  $\infty$  and  $p$  for any  $1 \leq m \leq n$  coprime to  $n$ . We may or may not include the fixed point  $p$  in the Cantor set. Apparently this gives  $2\varphi(n)$  different conjugacy classes in  $\text{Map}(S)$  by looking at the rotation number and whether the fixed point  $p$  lies in the Cantor set.

Conversely, suppose we have two homeomorphisms  $\tilde{g}_i$  on  $S^2$  as above fixing  $\infty$  and  $p_i \neq \infty$  such that either both  $p_i \in K$  or  $p_i \notin K$ ,  $i = 1, 2$ . Suppose further that they have the same rotation number  $m/n \pmod{\mathbb{Z}}$ . Let  $q_i : S^2 \rightarrow S^2/\tilde{g}_i$  be the quotient map. Then  $q_i(K)$  is still a Cantor set. There is a homeomorphism  $h : S^2/\tilde{g}_1 \rightarrow S^2/\tilde{g}_2$  taking  $q_1(K)$  to  $q_2(K)$  and  $q_1(\infty)$  to  $q_2(\infty)$ . Moreover, in the case  $p_i \in K_i$ , we can choose  $h$  so that  $h(q_1(p_1)) = q_2(p_2)$ . Then the map

$$h \circ q_1 : S^2 \setminus \{\infty, p_1\} \rightarrow (S^2/\tilde{g}_2) \setminus \{q_2(\infty), q_2(p_2)\}$$

lifts to  $S^2 \setminus \{\infty, p_2\}$ , which extends uniquely to a map  $\tilde{h} : S^2 \rightarrow S^2$ . The map  $\tilde{h}$  preserves the Cantor set  $K$ , satisfies  $\tilde{h}(\infty) = \infty$ ,  $\tilde{h}(p_1) = p_2$ , and fits into the following commutative diagram.

$$\begin{array}{ccc} S^2 & \xrightarrow{\tilde{h}} & S^2 \\ q_1 \downarrow & & q_2 \downarrow \\ S^2/\tilde{g}_1 & \xrightarrow{h} & S^2/\tilde{g}_2 \end{array}$$

For any  $x_0 \in S^2 \setminus \{\infty, p_1\}$ , let  $x_j = \tilde{g}_1^j x_0$  for  $0 \leq j \leq n-1$ . Fix a short ray  $r$  that passes through  $x_0$  but not any  $x_j$  for  $j \neq 0$  such that  $\{\tilde{g}_1^j r\}_{j=1}^{n-1}$  are disjoint (except at  $\infty$ ). Such a ray can be obtained for instance by lifting a short ray on  $S^2/\tilde{g}_1$ . Then there is a permutation  $\sigma$  on  $\{0, 1, \dots, n-1\}$  such that  $\tilde{h}(\tilde{g}_1^j x) = \tilde{g}_2^{\sigma(j)} \tilde{h}(x)$  for all  $x$  on  $r$  and all  $0 \leq j \leq n-1$ . Since  $\tilde{g}_1$  and  $\tilde{g}_2$  have the same rotation number and  $\tilde{h}$  maps  $\{\tilde{g}_1^j r\}_{j=1}^{n-1}$  to  $\{\tilde{g}_2^j \tilde{h}(r)\}_{j=1}^{n-1}$  preserving their circular order on the conical circle  $S_C^1$ , we must have  $\sigma = 1$  and  $\tilde{h}\tilde{g}_1(x_0) = \tilde{g}_2\tilde{h}(x_0)$ . Since  $x_0$  is arbitrary, we conclude that  $g_1$  and  $g_2$  are conjugate by the image of  $\tilde{h}$  in  $\text{Map}(S)$ .  $\square$

#### 4. USING TORSION TO OBSTRUCT EMBEDDINGS

If  $S$  is an orientable, infinite-type surface,  $\text{Map}(S)$  has a natural nontrivial topology called the *permutation topology* which agrees with the quotient topology inherited from  $\text{Homeo}(S)$ . Let  $\mathcal{C}(S)$  denote the set of isotopy classes of essential, simple closed curves in  $S$ . The set  $\mathcal{C}(S)$  is countable, and it follows from work of Hernández Hernández–Morales–Valdez [HHMV19] that the action of  $\text{Map}(S)$  on  $\mathcal{C}(S)$  is faithful. A neighborhood basis of the identity in  $\text{Map}(S)$  is given by the sets

$$\bigcap_{c \in C} \text{Stab}(c),$$

where  $C$  ranges over the finite subsets of  $\mathcal{C}(S)$ . The action of  $\text{Map}(S)$  on  $\mathcal{C}(S)$  exhibits  $\text{Map}(S)$  as a closed subgroup of  $\text{Sym}(\mathbb{N})$ , the group of bijections of a countable set, again given the permutation topology [Vla19, Corollary 6]. A natural question to ask is whether

$\text{Sym}(\mathbb{N})$  embeds in any big mapping class group. The main result of this section is that there is no embedding.

**Theorem 8.** *If  $G$  is a topological group containing a sequence of nontrivial torsion elements limiting to the identity, then  $G$  does not embed (as a topological group) in  $\text{Map}(S)$ , for  $S$  an orientable surface whose boundary is empty or a union of circles.*

Examples of such groups  $G$  include  $\text{Sym}(\mathbb{N})$ ,  $\text{Out}(\pi_1(S))$  when  $S$  is of infinite type, the automorphism group of a rooted tree, and  $\text{Homeo}(S^1)$ . Theorem 8 follows from the following proposition, which exhibits an open neighborhood of the identity in  $\text{Map}(S)$  that contains no nontrivial torsion elements.

**Proposition 9.** *Let  $\gamma_1, \gamma_2, \gamma_3$  be essential, simple closed curves that cobound a pair of pants. Any finite order element of*

$$\text{Stab}[\gamma_1] \cap \text{Stab}[\gamma_2] \cap \text{Stab}[\gamma_3]$$

*in the action of  $\text{Map}(S)$  on  $\mathcal{C}(S)$  is the identity.*

*Proof.* Suppose  $f \in \text{Map}(S)$  has finite order, and that  $f$  preserves the isotopy class of each  $\gamma_i$  as in the statement. By Theorem 2, there is a hyperbolic metric on  $S$  such that  $f$  may be represented by an isometry  $\varphi: S \rightarrow S$ . The isometry  $\varphi$  restricts to an isometry of the pair of pants  $P$  bounded by the geodesic representatives of the curves  $\gamma_i$ . Since  $f$  fixes the isotopy class of each  $\gamma_i$ , the isometry  $\varphi$  must fix each geodesic representative setwise. But this implies  $\varphi$  restricts to the identity on  $P$ , from which it follows that  $\varphi: S \rightarrow S$  is the identity.  $\square$

## 5. COMPACT SUBGROUPS ARE FINITE

The purpose of this section is to prove the following theorem.

**Theorem 10.** *Let  $S$  be an orientable surface whose boundary is either empty or a union of circles. Compact subgroups of  $\text{Map}(S)$  are finite, and locally compact subgroups are discrete.*

Of course, if  $S$  is a finite-type surface, the permutation topology on  $\text{Map}(S)$  is discrete, and there is nothing to prove. So suppose  $S$  is of infinite type. The permutation topology on  $\text{Sym}(\mathbb{N})$  is Hausdorff and totally disconnected [Cam96, p. 136], properties which pass to subspaces like  $\text{Map}(S)$ . A theorem of van Dantzig [Wes18, Theorem 1.3] says that totally disconnected, locally compact groups admit a neighborhood basis of the identity given by compact, open subgroups. Finite Hausdorff spaces are discrete, thus Theorem 10 reduces to showing that compact subgroups of  $\text{Map}(S)$  are finite.

Let  $K \leq \text{Map}(S)$  be a compact subgroup. We will show that  $K$  is virtually torsion-free and that every element of  $K$  has finite order. Only the trivial subgroup of such a group is torsion-free, thus it follows that  $K$  itself is finite.

Therefore, the proof of Theorem 10 reduces to the following two lemmas.

**Lemma 11.** *If  $S$  is an orientable, infinite-type surface, compact subgroups of  $\text{Map}(S)$  are virtually torsion-free.*

*Proof.* It is well-known that compact Hausdorff, totally disconnected groups are isomorphic as topological groups to *profinite* groups, inverse limits of finite groups. Open subgroups of profinite groups are of finite index. In fact, if  $G = \varprojlim_{i \in I} G_i$  is an inverse limit of the inverse system of finite groups  $G_i$ , then an open neighborhood basis of the identity in  $G$  is given by the kernels of the maps  $G \rightarrow G_i$  as  $i$  varies.

Thus if  $K$  is a compact subgroup of  $\text{Map}(S)$ , the intersection  $K \cap U$  of  $K$  with  $U \subset \text{Map}(S)$  any open neighborhood of the identity contains a finite-index subgroup of  $K$ . Examples of torsion-free neighborhoods of the identity were constructed in Proposition 9, establishing the lemma.  $\square$

**Lemma 12.** *If  $S$  is an orientable, infinite-type surface and  $K$  is a compact subgroup of  $\text{Map}(S)$ , every element of  $K$  has finite order.*

*Proof.* As discussed in the proof of the previous lemma, if  $K$  is a compact subgroup of  $\text{Map}(S)$  and  $U$  is an open neighborhood of the identity, the intersection  $K \cap U$  contains a finite-index subgroup of  $\text{Map}(S)$ . Let  $\gamma$  be the isotopy class of an essential, simple closed curve. An example of such an open neighborhood  $U$  is given by those elements of  $\text{Map}(S)$  preserving  $\gamma$ . If  $K'$  is a finite-index subgroup of  $K$ , every element  $g \in K'$  has a positive power lying in  $K'$ . Thus as  $\gamma$  varies, we see that  $g$  acts periodically on the isotopy class of every essential simple closed curve in  $S$ .

We claim that an element  $g$  of  $\text{Map}(S)$  which acts periodically on the isotopy class of every essential simple closed curve in  $S$  has finite order in  $\text{Map}(S)$ .

First suppose  $S'$  is a finite-type subsurface of  $\text{Map}(S)$ . We will show that a power of  $g$  fixes  $S'$  up to isotopy and induces the identity element of  $\text{Map}(S')$ . Indeed, the Alexander method [FM12, Proposition 2.8] for finite-type surfaces implies the existence of a finite collection  $C$  of essential, simple closed curves on  $S'$  such that any element of  $\text{Map}(S)$  fixing pointwise each element of  $C$  and fixing the isotopy class of each boundary curve of  $S'$  in  $S$  restricts to the identity element of  $\text{Map}(S')$ . An element  $g$  as above acts periodically on each isotopy class, thus a power  $g^k$  of  $g$  restricts to the identity element of  $\text{Map}(S')$ .

Thus if  $S'$  is a finite-type subsurface of  $S$  there exists a positive integer  $k$  such that  $g^k$  preserves  $S'$  up to isotopy and induces the identity element of  $\text{Map}(S')$ . Choose a diffeomorphism  $\varphi: S \rightarrow S$  representing  $g$  such that  $\varphi^k|_{S'}$  is the identity. Then

$$S'' = \bigcup_{i=1}^k \varphi^i(S')$$

is a finite-type subsurface of  $S$  which is preserved up to isotopy by  $g$ . Indeed, by induction as in Proposition 6 there is an exhaustion of  $S$  by finite-type subsurfaces  $\emptyset = S_0 \subset S_1 \subset \dots$  such that for each  $j$ ,  $S_j$  is  $g$ -invariant up to isotopy, and the restriction of  $g$  to  $\text{Map}(S_j)$  has finite order.

The final step is to show that  $g$  itself has finite order. Let  $k_j$  denote the order of  $g$  when restricted to  $\text{Map}(S_j)$ . The claim follows once we show that the orders  $k_j$  are uniformly bounded. Indeed, let  $S_i$  and  $S_j$  be terms in the invariant exhaustion above satisfying  $S_i \subset S_j$ . Suppose further that  $S_i$  contains three simple closed curves which cobound a pair of pants. As in Proposition 9, observe that any finite order element of  $\text{Map}(S_j)$  preserving the isotopy class of each of these three curves is the identity of  $\text{Map}(S_j)$ . Thus  $i \leq j$  implies the powers  $k_j$  and  $k_i$  satisfy  $k_j \leq k_i$ . Indeed, we conclude the order of  $g$  in  $\text{Map}(S)$  divides  $k_i$ .  $\square$

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