NOTES ON ONE-RELATOR QUOTIENTS

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Abstract. This is a summary of some known results/arguments about non-triviality of one-relator quotients and their generalizations.

1. The Kervaire conjecture

Conjecture 1.1 (Kervaire). *For any nontrivial group G and any $w \in G\star\mathbb{Z}$, the quotient $(G\star\mathbb{Z})/\langle\langle w\rangle\rangle$ is nontrivial.*

This also appears as Problem 5.7 in Kirby's (1990s) problem list [Kir97] contributed by Freedman. It is easy to prove by taking abelianization when the projection $p_{\mathbb{Z}}(w)$ of w to $\mathbb{Z}$ is not ± 1. So it reduces to the case where $p_{\mathbb{Z}}(w) = 1$. In this case, it suffices to prove the following stronger conjecture. Here $p_{\mathbb{Z}} : G \star \mathbb{Z} \to \mathbb{Z}$ is the projection to $\mathbb{Z}$.

Conjecture 1.2 (Kervaire–Laudenbach). *For any group G and any $w \in G \star \mathbb{Z}$ with $p_{\mathbb{Z}}(w) = 1$, the natural map $G \to (G \star \mathbb{Z})/\langle\langle w\rangle\rangle$ induced by the inclusion $G \to G \star \mathbb{Z}$ is injective.*

This fails if $p_{\mathbb{Z}}(w) = 0$, e.g. when $G = \mathbb{Z}/2 \star \mathbb{Z}/3 = \langle a\rangle \star \langle b\rangle$ with $w = atbt^{-1}$, where t is a generator of the $\mathbb{Z}$ factor.

A stronger statement is to only assume $p_{\mathbb{Z}}(w) \neq 0$.

Here are some known results on the Kervaire-Laudenbach conjecture.

Theorem 1.3 (Gerstenhaber–Rothaus [GR62], finite groups). *The Kervaire–Laudenbach conjecture holds for any finite group G under the weaker assumption $p_{\mathbb{Z}}(w) \neq 0$. In particular, the Kervaire conjecture holds for finite groups.*

Proof. Since G is finite, it embeds into a unitary group $U(n)$ for n large. Fix such an embedding $i : G \to U(n)$. The goal is to construct a homomorphism $\varphi : G \star \mathbb{Z} \to U(n)$ so that

(1) its restriction on the G factor is i, and
(2) $\varphi(w) = 1$.

The latter requirement ensures that φ induces a homomorphism $\bar\varphi : (G \star \mathbb{Z})/\langle\langle w\rangle\rangle \to U(n)$, whose pre-composition with the natural inclusion of the G factor is the embedding i, and thus this implies that the natural inclusion is injective as desired.

Any homomorphism φ satisfying the first requirement is uniquely determined by the image of t (the generator of $\mathbb{Z}$ with $p_{\mathbb{Z}}(t) = 1$). Suppose $w = \prod_j g_j t^{e_j}$ with $g_j \in G$ and $e_j \in \mathbb{Z} \setminus \{0\}$. Then it suffices to choose $x := \varphi(t) \in U(n)$ so that $y := \prod_j i(g_j)x^{e_j} = id \in U(n)$. Consider the map $f_w : U(n) \to U(n)$ with $f_w(x) = y$. We show f_w is surjective to conclude that there is some x so that $y = f_w(x) = id \in U(n)$, for which setting $\varphi(t) = x$ gives the desired homomorphism.

Since $U(n)$ is a closed orientable manifold, it suffices to show that f_w has nonzero degree. Since $U(n)$ is connected, choosing a path between id and $i(g_j)$ for each j gives a homotopy between f_w and g_w, where $g_w(x) := \prod_j id \cdot x^{e_j} = x^{\sum e_j} = x^{p_{\mathbb{Z}}(w)}$. It is known that g_w has nonzero degree as $p_{\mathbb{Z}}(w) \neq 0$, hence so does f_w. This completes the proof. □

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Corollary 1.4 (residually finite and hyperlinear)**.** *The Kervaire–Laudenbach conjecture holds for any residually finite group (resp. hyperlinear) G under the weaker assumption $p_{\mathbb{Z}}(w) \neq 0$. In particular, the Kervaire conjecture holds for these groups.*

Proof. Given any $g \neq id \in G$ with G residually finite, to show its image is nontrivial in $(G \star \mathbb{Z})/\langle\langle w \rangle\rangle$, take a homomorphism $\varphi : G \to F$ for some finite group F with $\varphi(w)$ nontrivial and apply the result for F.

For hyperlinear groups, I don't exactly know the proof, but roughly it goes like this: The defining property gives nice maps to (product of) unitary groups, and the result for finite groups are proved by embedding finite groups in unitary groups, so just use an analogous argument. Sofic groups and amenable groups are hyperlinear. □

Another important progress on the Kervaire–Laudenbach conjecture is done by Klyachko. A more careful explanation of Klyachko's argument can be found in [FR96].

Theorem 1.5 (Klyachko [Kly93], torsion-free groups)**.** *The Kervaire–Laudenbach conjecture holds for any torsion-free group G. In particular, the Kervaire conjecture holds for torsion-free groups.*

There are various generalizations of the results above under more technical assumptions on the groups or on w.

An unsolved folklore conjecture related to Klyachko's theorem is:

Conjecture 1.6. *For any torsion-free groups A and B, and any $w \in A \star B$ that does not conjugate into A, then the natural map $A \to (A \star B)/\langle\langle w \rangle\rangle$ is injective. As a consequence, for any $w \in A \star B$, the quotient $(A \star B)/\langle\langle w \rangle\rangle$ is nontrivial.*

This appears in Kirby's (1970s) problem list [Kir78] as Problem 66 (under the additional assumption that $H_1(A \star B)$ is 0 or $\mathbb{Z}$) contributed by Freedman.

Klyachko's theorem proves the case where $B = \mathbb{Z}$ (under additional assumptions on w). It is known if A and B are assumed to be locally indicable (meaning that any finitely generated nontrivial subgroup surjects $\mathbb{Z}$), which is first proved by Brodskiĭ [Bro84] using an algebraic argument, independently by Howie [How82] for a more general version (see Theorem [How82] below) using topological methods.

The conjecture above is also known when w is a (high) proper power (i.e. $w = u^k$ for some $k \geq 4$) by a theorem of Howie [How90].

2. Generalizations to free products with more factors and more relators

There is a generalization of the Kervaire–Laudenbach Conjecture 1.2 to the case of multiple relators that has attracted a lot of attention. This problem as well as the Kervaire–Laudenbach conjecture has been studied as *equations over groups*. Using this formulation, much more details can be found in the survey [Rom12] by Roman'kov.

Here is the setup. Let $X = \{x_1, \cdots, x_m\}$ be a finite set and $F(X)$ the free group with basis X. Consider $G_X := G \star F(X)$ and let $p_i : G_X \to \mathbb{Z}$ be the projection to the $\mathbb{Z}$ factor generated by x_i. Explicitly, p_i counts the exponent sum of the generator x_i. For m elements $w_1, \cdots, w_m \in G_X$, we form a $m \times m$ integral matrix M with entries $M_{ij} = p_i(w_j)$.

The most general conjecture regarding non-triviality of the quotient $G_X/\langle\langle w_1, \cdots, w_m \rangle\rangle$ is the following, also referred to as the Kervaire–Laudenbach Conjecture (or sometimes Howie's conjecture).

Conjecture 2.1. *For any group G and a set X of cardinality m, for any m elements $w_1, \cdots, w_m \in G_X$ with $\det(M) \neq 0$, the natural map $G \to G_X/\langle\langle w_1, \cdots, w_m \rangle\rangle$ is injective.*

The earlier Conjecture 1.2 is for the case $m = 1$ with the stronger assumption $\det(M) = 1$.

The theorem of Gerstenhaber–Rothaus (Theorem 1.3) actually generalizes to this setting without much difficulty.

Theorem 2.2 (Gerstenhaber–Rothaus [GR62], finite groups)**.** *For any G finite and a set X of cardinality m, for any m elements $w_1, \cdots, w_m \in G_X$ with $\det(M) \neq 0$, the natural map $G \to G_X/\langle\langle w_1, \cdots, w_m\rangle\rangle$ is injective. In particular the quotient is nontrivial if G is.*

For torsion-free groups, the theorem of Klyachko (Theorem 1.5), however, remains as an solved conjecture. The most general version of it is attributed to Levin.

Conjecture 2.3 (Levin)**.** *For any torsion-free group G and a set X of cardinality m, for any m elements $w_1, \cdots, w_m \in G_X$ (no matter what $\det(M)$ is), the natural map $G \to G_X/\langle\langle w_1, \cdots, w_m\rangle\rangle$ is injective.*

On the other hand, if one assumes the stronger assumption that G is locally indicable (meaning that every nontrivial finitely generated subgroup surjects $\mathbb{Z}$), then we have the following theorem of Howie [How82].

Theorem 2.4 (Howie)**.** *For any locally indicable group G and a set X of cardinality m, for any m elements $w_1, \cdots, w_m \in G_X$ with $\det(M) \neq 0$, the natural map $G \to G_X/\langle\langle w_1, \cdots, w_m\rangle\rangle$ is injective.*

There is a mod p version of this by Gersten [Ger87], which has an alternative proof by Krstić [Krs85]. Also see [Rom12, Theorem 2.5] for the formulation.

If one still takes a *one-relator* quotient of a free product with more than two factors, then it is conjectured that the result is nontrivial. This appears in [Gor83, Conjecture 9.5].

Conjecture 2.5. *If $G_i \neq id$ for $i = 1, 2, 3$, then $(G_1 \star G_2 \star G_3)/\langle\langle w\rangle\rangle \neq id$ for any $w \in G_1 \star G_2 \star G_3$.*

Actually, it is conjectured that at least one of the three factors will inject [Gor83, Conjecture 9.4]. This conjecture implies the three summand conjecture about Dehn surgery of knots. The special case with each G_i cyclic is known as the Scott–Wiegold conjecture, proved by Howie [How02]. It seems that very little is known beyond this.

3. Additional details

Suppose $\mathcal{G}$ is a class of groups that are closed under taking free products. Then Conjecture 1.2 restricted to $\mathcal{G}$ reduces to the case where one factor is $\mathbb{Z}$. So they are equivalent if $\mathcal{G}$ contains $\mathbb{Z}$.

Proposition 3.1. *Let $\mathcal{G}$ be a class of groups closed under taking free products. Suppose $G \to (G \star \mathbb{Z})/\langle\langle w\rangle\rangle$ is injective for all $G \in \mathcal{G}$ and any $w \in G \star \mathbb{Z}$ not conjugate into G with $p_{\mathbb{Z}}(w) = 0$ (or even more specifically, see the proof below). Then $A \to (A \star B)/\langle\langle w\rangle\rangle$ is injective for all $w \in A \star B$ not conjugate into A for any $A, B \in \mathcal{G}$.*

Proof. Let $G = A \star B$, which is in $\mathcal{G}$ since both A, B are. The case where $w \in A \star B$ conjugates into B is obvious. Now suppose $w = a_1 b_1 \cdots a_k b_k$ as a cyclically reduced word for some $k \geq 1$. Let $w(t) = a_1 t b_1 t^{-1} \cdots a_k t b_k t^{-1} \in G$.

Consider the embedding $f : A \star B \to A \star B \star \langle t\rangle$ given by $f(a) = a$ and $f(b) = tbt^{-1}$ for all $a \in A$ and $b \in B$. Then $f(w) = a_1 t b_1 t^{-1} \cdots a_k t b_k t^{-1}$ has t-exponent sum zero. There is an induced map $\bar{f} : (A \star B)/\langle\langle w\rangle\rangle \to (A \star B \star \langle t\rangle)/\langle\langle f(w)\rangle\rangle$ that makes the following diagram commute:

$$\begin{tikzcd} A \star B \arrow[r, "f"] \arrow[d, "q_1"] & A \star B \star \langle t\rangle \arrow[d, "q_2"] \\ (A \star B)/\langle\langle w\rangle\rangle \arrow[r, "\bar{f}"] & (A \star B \star \langle t\rangle)/\langle\langle f(w)\rangle\rangle. \end{tikzcd}$$

By assumption, q_2 restricted to the factor $G = A \star B$ is injective. Note that f restricted to A is its standard embedding (and lands in the factor $A \star B$), so $q_2 \circ f|_A$ is also injective. By commutativity of the diagram, $q_1|_A$ is injective as desired. $\square$

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